The Ultralong Sovereign Default Risk

Radoslaw Paluszynski*

Department of Economics
University of Houston

September 1, 2024

click here for most recent version

Abstract

Between 2010 and 2015, Mexican government issued external debt worth 0.5% of its GDP in the form of century bonds. Using a sovereign default model with endogenous maturity and variable risk-free rate, I propose a theory of the ultralong debt issuances and investigate the resulting bond spreads. Government issues such bonds in order to insure against low-frequency movements in the risk-free rate, and the benefit from such hedging is largest when interest rates are low. The model calibrated to Mexico's default history comes remarkably close to predicting the spreads on 100-year bonds observed in the data. This suggests that Mexico is expected to remain a frequent defaulter in the next century.

Keywords: Sovereign default, endogenous debt maturity

JEL Classification Numbers: F34, G12, G15

*E-mail: rpaluszynski@uh.edu. This paper has benefited from very helpful comments by Manuel Amador, Cristina Arellano, Klaus Desmet, Hyunju Lee, Leo Martinez, George Stefanidis, Kei-Mu Yi; seminar participants at Ryerson University, Southern Methodist University, University of Queensland, International Monetary Fund; and conference participants at the 2022 Summer Workshop on Macroeconomics and Finance in Warsaw, 2022 Leuven Summer Event, 2021 SED, 2019 NBER IFM Summer Institute, 2019 SAET, 2018 Fall Midwest Macro meetings. The quantitative part of this paper was conducted using the resources of the Core facility for Advanced Computing and Data Science (CACDS) at the University of Houston.

1 Introduction

This paper analyzes the recent emergence of a rare security in public finance, the so-called sovereign 100-year bonds. Since 2010, a few governments around the world have auctioned bonds that will mature in the 22nd century. Such long-dated instruments are not entirely new in the financial markets - the UK government used to issue perpetuities until the early 1900s (some of which were only bought back in 2015), while large American corporations such as IBM, Coca-Cola and Walt Disney launched their own century bonds in the late 1990s. What is unusual about the most recent wave is that such bonds were also issued by emerging markets notorious for sovereign defaults, most notably Mexico.

In years 2010 to 2015, the Mexican government issued debt worth around 0.5% of its GDP in foreign-denominated century bonds. In the one hundred years prior to 2010, Mexico's government defaulted on external creditors three times.[1] Yet, the common perception among the financial markets commentators was that Mexico secured unusually favorable terms of these bond issuances, with coupon rates ranging from 4% to 5.75%. In its coverage of the topic in 2015, the Economist wrote:[2] *Foreign creditors are more bullish (...) Those are extraordinarily good terms given Mexico's distinctly spotty credit record.* This provokes natural research questions: what has triggered the recent emergence of sovereign century bonds? Are the interest rates that emerging markets pay on these bonds consistent with their history of defaults? And if not, then what can we infer from observing the prices of such securities about the markets' expectations of Mexico's creditworthiness in the next one hundred years?

To answer these questions, I build a quantitative model of sovereign default with endogenous choice of government debt maturity. The model is similar to the ones of Arellano and Ramanarayanan (2012) and Hatchondo, Martinez and Sosa-Padilla (2016), except that I assume both types of debt to be long-term. The first type represents a general stock of government's external debt and is calibrated to match its average size and maturity. The second type is a perpetuity, i.e. bond that pays a fixed coupon forever, or until it is repurchased. The government faces a stochastic stream of GDP growth rates and lacks commitment to honor its obligations. A non-repayment puts both types of debt uniformly in the state of default, and bond yields compensate the foreign lenders for this risk. The optimal maturity structure is then jointly determined in equilibrium along with bond prices.

[1]See Reinhart and Rogoff (2009). This does not include the debt crisis of 1994/95 in which a default was averted only as a result of the bailout provided by the US government and the IMF.

[2]"The 100-year view", The Economist, May 2nd 2015.

The model departs from standard frameworks in that it introduces low-frequency variations in the risk-free rate over time. Consistent with historical data, the long-term real interest rate follows an autoregressive process ranging between 8% in the 1980 and -1% in 2012, and is projected to converge to some level around 4.5% in the next 50 years, according to the available long-horizon forecasts (which I document in Section 2.3). This uncertainty about the future movements in the risk-free rate provides a motive to use the ultralong debt to hedge against them. A positive shock to the interest rate tomorrow implies that debt will be more expensive and the impatient borrower's marginal utility will increase. However, a higher interest rate also causes all bond prices and the value of all outstanding debt to drop, lowering the total debt burden of the government. As a result, by going long the government can insure against future changes in the risk-free rate. But because these movements tend to be very persistent, the government needs a debt instrument that extends far beyond the maturity of typical long-term bonds. An ultralong bond, modeled here for simplicity as a perpetuity, incorporates the entire expected path of future interest rate shocks and can thus be used effectively to hedge against them.

At the same time, however, issuing perpetuities is costly to the government because such bonds provide fewer repayment incentives next period and promote more debt dilution. By selling ultralong bonds, the government faces steeper declines in the prices of all bonds due to the fact that its future self will be less concerned about rolling them over and securing high prices. In equilibrium, the government chooses an optimal mix of bonds with regular maturity and perpetuities by trading off the benefits from hedging with an attempt to provide itself with enough incentives to maintain high bond prices in next period. Previous literature has shown that the motive to hedge against income shocks is sufficient to generate realistic levels of long-term bonds with regular maturity, but not necessarily for longer-dated instruments. The main insight of this paper is that the emergence of ultralong bonds can be rationalized by focusing instead on persistent shocks to the risk-free rate.

In the simulation exercises, I find that the government calibrated to Mexico decides to issue relatively small amounts of the ultralong debt, with a share of 2-8% in total debt, which is remarkably close to what we observed in the data. Crucially, such issuances occur predominantly in the states where the risk-free rate is low. This is because when interest rate falls, an impatient borrower is inclined to take on more debt and puts himself at a higher risk of default due to a looming possibility of an interest rate increase in the future. This causes future bond prices and the government's marginal utility of consumption to fluctuate more, increasing the gains from hedging against such shocks. By contrast, when the risk-free rate is high, debt is reduced and default probability declines as the future interest is bound to mean-revert downwards. In such states, the potential benefit from hedging is reduced as the future bond prices and marginal utility exhibit smaller volatility and comove less. Indeed, the simulated model shows that the average welfare gain from being able to issue ultralong bonds is equivalent to 0.4% of current consumption when the (real) risk-free rate is 1%, but it declines to around 0.05% of current consumption when the risk-free rate is 8%.

To analyze the spreads on 100-year bonds observed in the data, I make the model more realistic by assuming that foreign lenders are risk-averse. Investor risk aversion has been shown to contribute to the sovereign spreads significantly, for example by Borri and Verdelhan (2011), Longstaff et al. (2011) and more recently Tourre (2017). I specify the pricing kernel directly using a two-factor affine term structure model and calibrate it to match the behavior of the US yield curve. Consistent with evidence, the risk-free yield curve in the model is upward-sloping on average due to the fact that shocks to the interest rate are correlated negatively with the state of US fundamentals, which influences the investors' marginal utility. High realizations of US fundamentals result in a more steeply upward sloping yield curve and partly undo the potential benefits of hedging with long-term debt.

Although investor risk aversion goes against the main mechanism presented in this paper, I find that the government still issues realistic amounts of the ultralong debt, especially so when the risk-free rate is low and foreign fundamentals are weak. I then use this augmented model to analyze the spreads on Mexico's 100-year bonds in years 2010-2015. I select a state characterized by the average amounts of outstanding debt, income growth and interest rates, as well as the level of foreign fundamentals similar to the ones in data over that period. The model predicts an issuance of roughly 7% share of the ultralong bonds in the total debt. Due a very low level of the risk-free rate, the model predicts steeply upward sloping yield curves for both the US and Mexico's bonds. Importantly, the difference between them, the spread curve, is itself upward sloping as in the data due to relatively low debt and high growth. At the long end of that curve, I find that the model *underpredicts* the spread on 100-year bonds by 0.2 percentage point relative to the available evidence. While there are various omitted factors in my analysis that may be important in the pricing of sovereign century bonds, I do not find support for a popular view that Mexico secured unusually favorable borrowing terms relative to its credit record.

1.1 Literature review

This paper is closely related to the quantitative sovereign default literature, in particular one building on the seminal works of Eaton and Gersovitz (1981), Aguiar and Gopinath (2006) and Arellano (2008). More recently, several papers have successfully introduced long-term debt models, starting with Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012), which is the main subject of my analysis.

My work relates to the branch of sovereign default models with endogenous maturity. In an important paper, Aguiar et al. (2019) show that in a basic model with maturity choice, governments should remain passive in the markets for long-term bonds in order to maximize the incentives for repayment next period. Arellano and Ramanarayanan (2012) or Hatchondo, Martinez and Sosa-Padilla (2016) demonstrate that this need not be case as in reality governments also seek to hedge against income shocks. In this paper, I further explore the hedging motive by showing that governments may in fact want to issue perpetual bonds in the presence of low-frequency fluctuations in the risk-free interest rate.

This paper also adds to an emerging branch of sovereign default models that incorporate shocks to the risk-free interest rate. Guimaraes (2011) shows that such shocks may be a more important factor driving sovereign defaults than fluctuations in output. Almeida et al. (2019) specifically ask whether the "Volcker shock" of 1981 caused the sequence of sovereign defaults in Latin America. They find that such causality could operate through an improvement in the borrowers' renegotiating position rather than through the direct increase in borrowing costs. Johri, Khan and Sosa-Padilla (2022) incorporate shocks to the level and volatility of the risk-free rate and show that they significantly increase the model's predictions for mean and variance of sovereign spreads. Tourre (2017) incorporates shocks to the risk-free rate in a continuous-time default model and shows that they explain the behavior of emerging markets' current account balance before and after 1980. In relation to these paper, I show that movements in the risk free rate on *low frequency* also have a potential to drive the maturity structure of sovereign debt.

The remainder of this paper is structured as follows. Section 2 discusses the evidence behind recent issuances of the ultralong debt. Section 3 introduces the main model. Section 4 calibrates the model and describes the basic mechanics. Section 5 conducts the simulation exercises and analyzes the spreads observed in the data. Section 6 augments the model with risk-averse lenders and Section 7 uses it to study Mexico's issuances in 2010-201. Section 8 discusses various extensions. Section 9 concludes.

2 Empirical analysis

In this section, I document the recent issuances of ultralong maturity debt by sovereign states, with an emphasis on the largest ones by Mexico. I further present evidence on the long-run expectations of the risk-free interest rate and Mexico's economic growth.

2.1 Sovereign century bonds

Table 1 collects evidence on the century bonds issued by sovereign states since the 1990s.[3] The market was opened by China and Philippines in 1996-1997 and continued until (most recently) Austria auctioned its century bond in 2017. Most recently, Israel and Peru joined the ranks of century bond issuers in 2020. While for most countries these bonds constitute a negligible percentage of total external debt, they are quite significant in the case of Mexico. In particular, notice that Mexico had three separate issuances between 2010 and 2015, each denominated in a different (foreign) currency and amounting to around 4.3% of Mexico's total external debt in 2010.

Table 1: Recent issuances of the century sovereign bonds

Country	Time	Currency	Coupon	Amount	% ext. debt
China	*Jan*.1996	USD	9.0	100	0.08
Philippines	*Jun*.1997	USD	8.6	100	0.20
Mexico	*Oct*.2010	USD	5.75	2,678	2.70
Mexico	*Mar*.2014	GBP	5.625	1,000	0.82
Mexico	*Apr*.2015	EUR	4.0	1,500	0.80
Belgium	*Sep*.2015	EUR	2.5	75	0.03
Belgium	*May*.2016	EUR	2.3	100	0.05
Ireland	*Apr*.2016	EUR	2.35	100	0.07
Argentina	*Jun*.2017	USD	7.125	2,750	2.16
Austria	*Sep*.2017	EUR	2.1	3,500	1.90
Israel	*Mar*.2020	USD	4.5	1,000	3.45
Peru	*Dec*.2020	USD	3.23	1,000	5.27

Source: Bloomberg. Issue amounts are in millions. Most bonds where issued at par which makes their coupon rates somewhat informative of the yield to maturity at issuance.

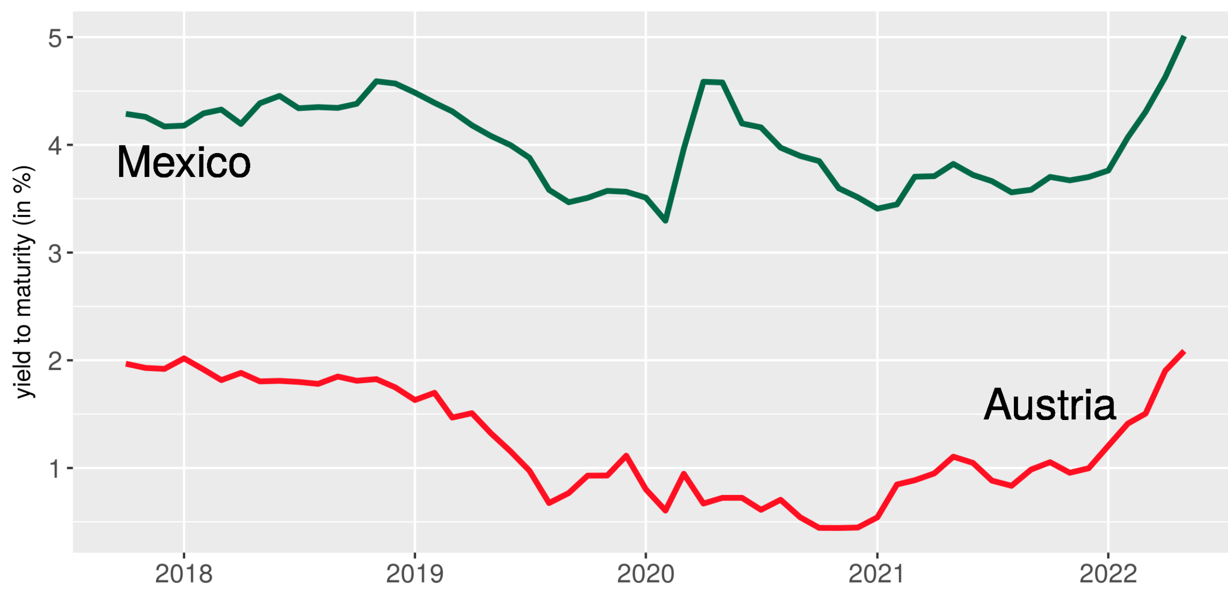
It is worth pointing out that the emergence of century bonds aligns with a recent trend of government extending the maturity profiles of their debt, as documented by Chen

[3] This list is not exhaustive going back longer in history. Up until the early 20th century perpetuities (ie. bonds without a specified maturity date) used to be a popular instrument used by governments. The most notable issuer was the UK government who bought back the last outstanding wartime consols in 2015.

et al. (2018). However, because the availability of data on sovereign debt maturity is limited, and because sovereign default models with a general maturity choice tend to be intractable, this paper is focused on the topic of 100-year bonds.

2.1.1 The case of Mexico

To shed more light on how the markets perceive Mexico's century bonds, Figure 1 plots the yields on Mexican EUR-denominated century bond, relative to the yields on Austria's 100-year bond which I assumed to be risk-free.[4] The reason I use Euro-denominated yields to measure the spread is that the US Department of Treasury does not issue bonds with maturity longer than 30 years. With the exception of the brief COVID-19 pandemic period in 2020, the spread on Mexico's century bond has been rather stable over time, fluctuating between 2 and 2.5 percent.



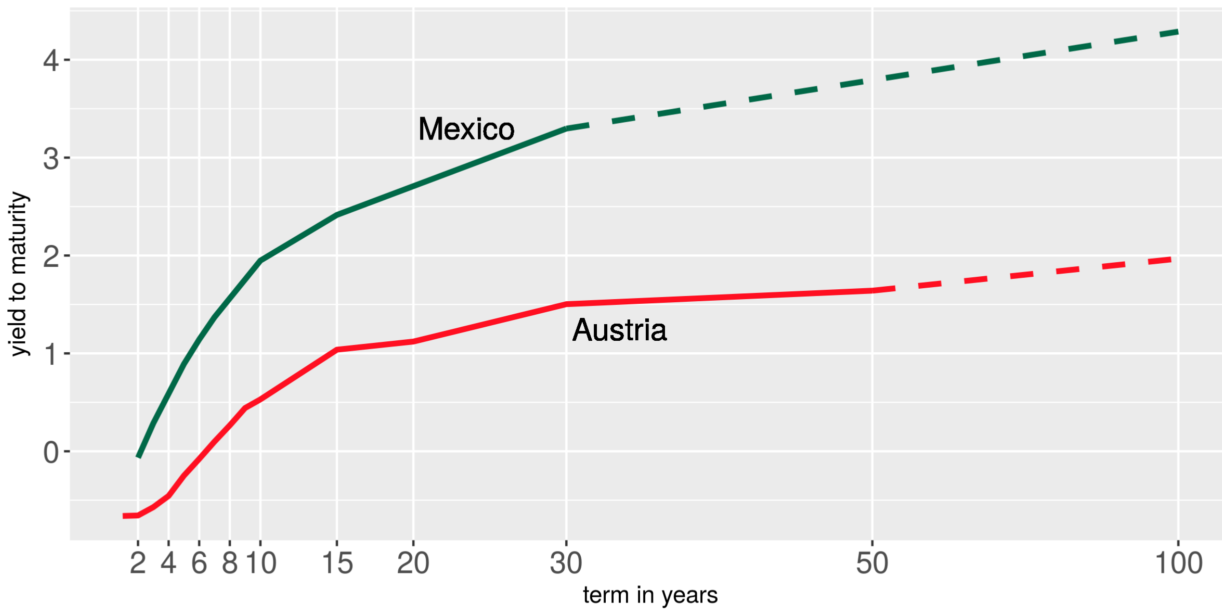
Source: Bloomberg. Time span: December 2012 - October 2017 (monthly averages).

Figure 1: Yields on Mexican and Austrian EUR-denominated century bonds over time

To get a sense of how the spread on Mexico's 100-year bonds compares to that for bonds of more common maturities, Figure 2 depicts the yield curve for Mexico's Euro-denominated bonds at a fixed point in time, relative to that of Austria. Notice that the spread curve is an increasing function of maturity: spread is below 1% for short-term debt, 1.5% for 10-year bonds, 1.8% for 30-year bonds, and finally 2.3% for 100-year bonds. One caveat in

[4]The yields on Austria's 100-year bonds are very close to those of Belgium and Ireland.

interpreting the curve in Figure 2 is that it describes the period of quantitative easing of the European Central Bank which depressed the yields on most Euro-denominated bonds, possibly affecting spreads as well. Nonetheless, it is probably the best available measurement of the spread for Mexico's 100-year bonds.



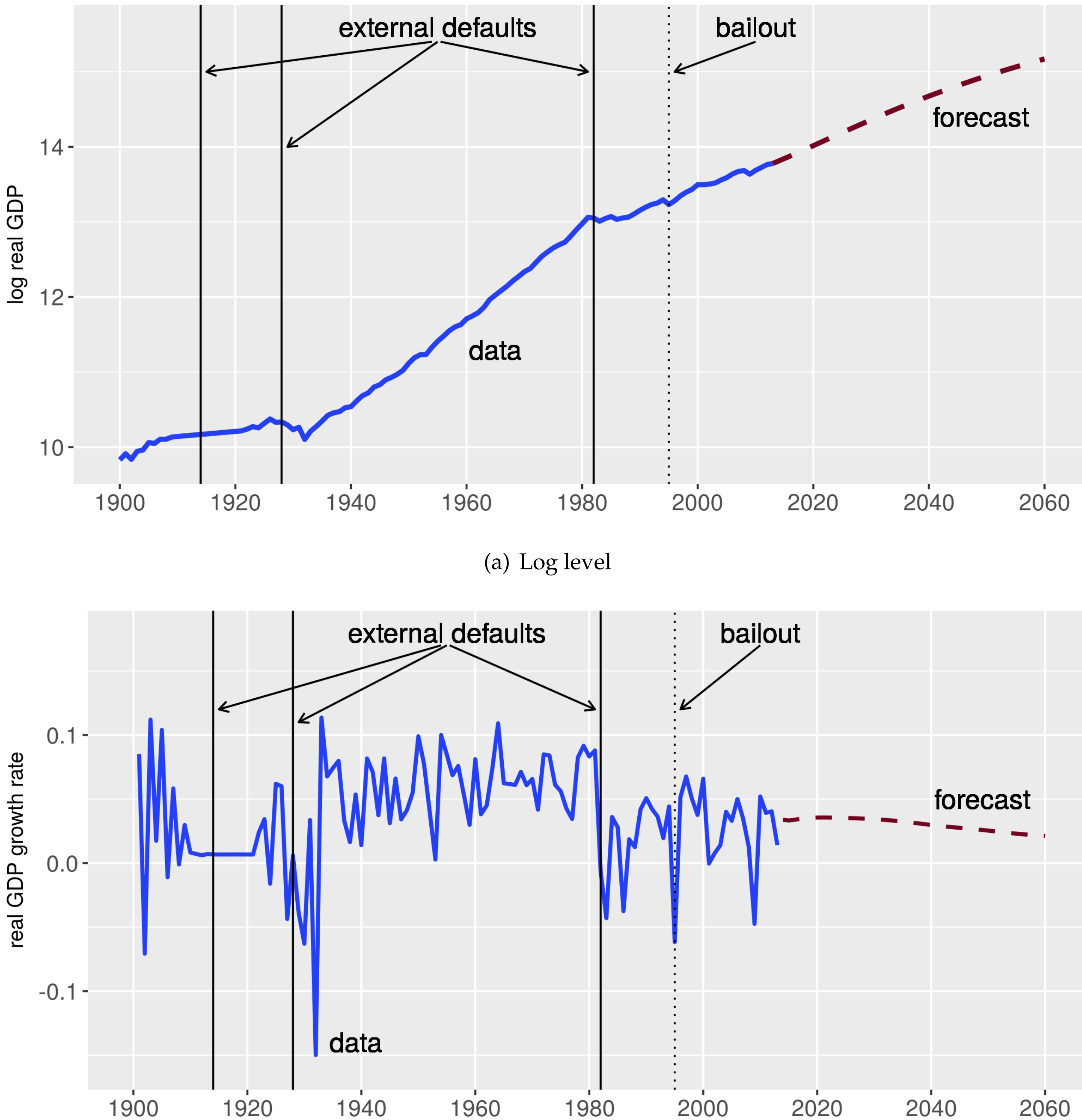
Source: Bloomberg, average yields for October 2017. The dashed part of each line represents interpolation over all the maturities between the 100-year bond and the longest among the shorter-maturity outstanding bonds.

Figure 2: Mexico vs. Austria yield curves

2.2 Mexico's economy in the long run

Figure 3 presents a history of Mexico's economic growth and sovereign defaults starting from the beginning of 20th century and including the OECD forecasts going all the way to 2060. The first thing to notice is that Mexico defaulted on its external creditors three times over the course of 100 years (during the 1995 tequila crisis default was avoided only as a result of the bailout provided by the US government and the IMF). Secondly, notice that Mexico's growth rate seems to follow two regimes, a high-average and low-volatility one (from mid 1930s until 1981) and a low-average and high-volatility regime that was in place before and after that period.[5] Debt crises have tended to occur in exclusively in the times of high volatility. The OECD forecasts that the country will be growing at a

[5] Specifically, the average and standard deviation of growth rate was 5.94% and 2.27%, respectively, in years 1934-1981; while it was 2.27% and 3.29%, respectively, in years 1982-2012.



(a) Log level

(b) Growth rate

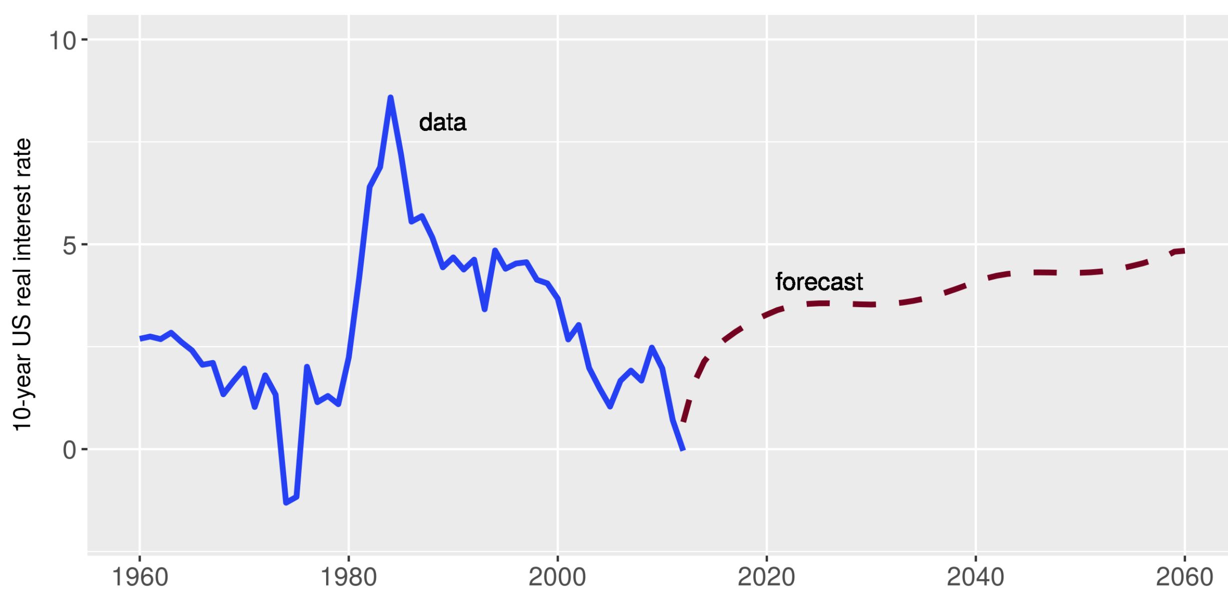
Source: INEGI (1995) and OECD. Solid line depicts Mexico's real GDP 1900-2012, dashed line shows the long-term projection published by OECD in June 2012. Note that the data is linearly interpolated for years 1910-1921 due to missing observations. The dates of external defaults are taken from Reinhart and Rogoff (2009) and cross-checked with Standard & Poor's (2014).

Figure 3: Mexican real GDP in the long run

slightly higher mean rate, but far below the pre-1981 average. Crucially, the long-term projections come without confidence intervals and there is no forecast of the volatility of growth which is a key parameter that determines the frequency of defaults. A natural question to ask then is whether the spreads on 100-year bonds can tell us anything about Mexico's growth volatility expected by the financial markets in next several decades.

2.3 Risk-free interest rate in the long run

An important factor in the choice of debt maturity is not just the spread over a risk-free rate (as it is commonly assumed by sovereign debt models), but also its level. Figure 4 shows that the real risk-free interest rate (measured by the 10-year US Treasury yield net of the CPI inflation rate) has been volatile since 1960, first increasing from below five to just short of ten percent in the early 1980s, and then gradually declining to zero in 2012. Importantly, the Figure shows that risk-free interest rate was then expected to recover to a long run level of around five percent (as proxied by the OECD's 2012 Long-Term Baseline Projection).



Source: OECD. Solid line depicts the yield on ten-year Treasury Bonds 1960-2012, dashed line shows the long-term projection published by OECD in June 2012.

Figure 4: Risk-free interest rate in the long run

A natural question concerns the accuracy and representativeness of OECD forecasts, especially at the 50-year horizon. While such long-term projections are extremely uncertain, what matters for the analysis in this paper is that they can be used as proxy for market-wide expectations. Indeed, Paluszynski (2023) shows that OECD forecasts for GDP growth during the European debt crisis are very similar to those of various private and public institutions. It is also important to notice that while more recent forecasts may be available, a 2012 one is particularly useful as it fall in the middle of the period of issuing the ultralong bonds by Mexico's government.

3 Model

In this section, I present the main environment of my analysis which features stochastic processes for the country's income and the risk-free interest rate.

3.1 Economic environment

Consider a representative-agent small open economy with a benevolent sovereign government that borrows internationally from a large number of competitive lenders. Time is discrete and a period is equal to one year. There is no production or labor. Instead, the economy faces a stochastic stream of endowment realizations. Markets are incomplete and the government has access to two non-contingent bonds of differing maturity.

Endowment process As in Aguiar and Gopinath (2006), the economy follows a stochastic trend and in each period the endowment realization is

$$log\, Y_t = \sum_{s=1}^{t} g_s \tag{1}$$

The growth rate g_t is assumed to be *i.i.d.*[6]

$$g_t = \mu + \sigma \varepsilon_t \tag{2}$$

where $\varepsilon_t \sim \mathcal{N}(0,1)$ is an *i.i.d.* random shock and $\{\mu, \sigma\}$ are the mean and variance parameters, respectively. The assumption of no persistence in growth rates is not crucial for the results. Instead, it allows me to reduce the computational burden of the model and to focus on the effects of introducing time variation in the risk-free interest rate.

[6] Although Aguiar and Gopinath (2006) argue that emerging market economies exhibit persistent shocks to trend growth, the Mexican data does not provide a strong support for it. Estimating equation (2) with lagged growth rate yields low and statistically insignificant estimates of the persistence parameters, especially if the most recent data is included.

Interest rate process The risk-free interest rate is assumed to follow an AR(1) process

$$r_{t+1} = (1-\rho_r)\bar{r} + \rho_r r_t + \sigma_r \varepsilon_{r,t+1} \tag{3}$$

where $\varepsilon_{r,t} \sim \mathcal{N}(0,1)$ is an *i.i.d.* random shock and $\{\bar{r}, \rho_r, \sigma_r\}$ are the mean, persistence and variance parameters, respectively.

Preferences The representative household has preferences given by the expected utility of the form

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t) \tag{4}$$

where I assume the function $u(\cdot)$ is strictly increasing, concave and twice continuously differentiable. The discount factor is given by $\beta \in (0,1)$.

Government In each period, the government chooses a consumption rule and a portfolio of foreign-denominated bonds to maximize the household's lifetime utility. The government may construct its portfolio using two types of debt. On the one hand, it has access to zero-coupon bonds of differing maturities $n \geq 1$ with face values $\{b_t^{(n)}\}$. For computational tractability, I restrict attention to the portfolios of bonds that satisfy $b_{n,t} = (1-\delta)^{n-1}B_t$ for all $n \geq 1$, where B_t is a scalar variable and δ is a parameter. This assumption allows me to collapse the entire portfolio of zero-coupon bonds into a single state variable as in Chatterjee and Eyigungor (2012) or Hatchondo and Martinez (2009).[7] On the other hand, the government can issue coupon-bearing perpetuities $B_{u,t}$ which do not have a maturity date but can be bought back in the future at current market price.

Default If the government borrows, it is not committed to repay the debt next period. Consequently, the bond is priced endogenously by risk-neutral lenders to account for the possibility of default as well as debt dilution in the future. I rule out the possibility of selective default, that is the government can only default on both types of bond simultaneously. As it is commonly assumed in the sovereign debt literature, the government who refuses to honor its obligations faces an exogenous cost of default and is further excluded from borrowing in the financial markets, with a certain probability of being readmitted in every subsequent period. When that happens, the government returns to credit markets with the amount of debt reduced by the "haircut" parameter $1-\omega$ (uniform across both debt instruments). While in default, the outstanding debt obligations grow at the current

[7] Appendix B formally shows this equivalence

rate of the international risk-free interest.

Market clearing There is no storage technology and, under the aforementioned assumptions on the utility function, implies that the endowment is fully divided between current consumption and net borrowing. This market clearing condition is given by

$$C_t = Y_t - B_t - B_{u,t}\kappa + q_t\big(B_{t+1} - (1-\delta)B_t\big) + q_{u,t}\big(B_{u,t+1} - B_{u,t}\big) \tag{5}$$

where $\{B_t, B_{u,t}\}$ are quantities of the two bond portfolios, $\{q_t, q_{u,t}\}$ are their prices, while κ is the coupon rate on the ultralong bond. Parameter δ represents the probability of maturing for a unit of the main bond B_t which can be thought of as an aggregation of the country's entire stock of "regular" maturity debt. The other bond, $B_{u,t}$, which can be associated with the ultralong bond, is for simplicity modeled as a perpetuity that does not mature but it can always be bought back.

Bond prices International lenders are perfectly competitive and have "deep pockets" in the sense that potentially even large losses do not affect their decisions. In equilibrium the lenders make expected zero profit and as a result, the bond pricing formula compensates them only for the default risk implied in the government's decisions.

3.2 Timeline

Timing of the model is standard. At the beginning of every period, shocks for the new endowment growth rate and the new risk-free interest rate are drawn. A government that is current on its obligations observes them and decides to repay or default on its debt. In the case of repayment, it simultaneously chooses new levels of borrowing in each bond, while international credit markets price these assets competitively based on the government's choices. In the case of default, the government is temporarily excluded from further borrowing, while international creditors swap their assets to defaulted bonds.

A government that enters the period excluded from the markets rolls a dice in an attempt to regain access. If it fails, the government stays in exclusion for the current period and its debt obligations grow at the risk-free interest rate. If it succeeds, a predetermined haircut amount is applied to its outstanding face value of debt, and the government can decide whether to honor the new obligations or default again.

3.3 Recursive formulation

In this section I formalize the economic environment by stating the problem faced by market participants in recursive form. To begin, I define the vector of exogenous state variables as $\mathbf{S} = (Y_{-1}, g, r)$, where g and r are the current growth rate and current interest rate, respectively, while Y_{-1} is last period's aggregate output level.[8] The endogenous state variables are B and B_u, the outstanding stocks of two types of sovereign debt.

Government The government that is current on its debt obligations has the general value function given by

$$v(B, B_u, \mathbf{S}) = \max_{d \in \{0,1\}} \Big\{ (1-d) v^r(B, B_u, \mathbf{S}) + d v^d(B, B_u, \mathbf{S}) \Big\} \tag{6}$$

A sovereign who defaults ($d = 1$) is excluded from international credit markets and has probability θ of being readmitted every subsequent period. The value associated with default is

$$\begin{aligned} v^d(B, B_u, \mathbf{S}) &= u\Big(Y(1-\phi)\Big) \\ &+ \beta \mathbb{E}\Big[\theta \underbrace{v\big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\big)}_{\text{value if readmitted}} + (1-\theta) \underbrace{v^d\big((1+r)B, (1+r)B_u, \mathbf{S}'\big)}_{\text{value if stays in exclusion}} |\mathbf{S}\Big] \end{aligned} \tag{7}$$

Defaulting on foreign debt is associated with two types of punishment. First, the government is excluded from financial markets and must live in autarky until it is stochastically readmitted. Second, the economy loses a fraction ϕ of its endowment every period in which the government stays in exclusion.

The value of the government associated with repayment of debt is given by

$$v^r(B, B_u, \mathbf{S}) = \max_{C, B', B'_u} \Big\{ u(C) + \beta \mathbb{E}\Big[v(B', B'_u, \mathbf{S}') | \mathbf{S} \Big] \Big\} \tag{8}$$

subject to the budget constraint formula

$$C = Y - B - B_u \kappa + q(B', B'_u, \mathbf{S})\Big(B' - (1-\delta)B\Big) + q_u(B', B'_u, \mathbf{S})\Big(B'_u - B_u\Big) \tag{9}$$

Having characterized the two value functions of the government, it is straightforward to

[8] Appendix A describes the model in detrended form that is used for computation.

derive the optimal decision rule for default as a function of today's state variables

$$d(B, B_u, \mathbf{S}) = \begin{cases} 1, & \text{if } v^d(B, B_u, \mathbf{S}) > v^r(B, B_u, \mathbf{S}) \\ 0, & \text{if } v^d(B, B_u, \mathbf{S}) \leq v^r(B, B_u, \mathbf{S}) \end{cases} \tag{10}$$

International Lenders The lenders post bond prices by discounting future cash flows at the current risk-free rate and taking the default risk explicitly into account. The price of the regular maturity debt is

$$\begin{aligned} q(B', B'_u, \mathbf{S}) = \frac{1}{1+r} \mathbb{E}\Big[& \underbrace{d\left(B', B'_u, \mathbf{S}'\right) \, q^d\left(B', B'_u, \mathbf{S}'\right)}_{\text{default}} \\ & + \underbrace{\Big(1 - d(B', B'_u, \mathbf{S}')\Big)\Big[1 + (1-\delta) q(B'', B''_u, \mathbf{S}')\Big]}_{\text{repayment}} \mid \mathbf{S} \Big] \end{aligned} \tag{11}$$

while the price of the ultralong bond is analogously given by

$$\begin{aligned} q_u(B', B'_u, \mathbf{S}) = \frac{1}{1+r} \mathbb{E}\Big[& \underbrace{d\left(B', B'_u, \mathbf{S}'\right) \, q^d_u\left(B', B'_u, \mathbf{S}'\right)}_{\text{default}} \\ & + \underbrace{\Big(1 - d(B', B'_u, \mathbf{S}')\Big)\Big(\kappa + q_u(B'', B''_u, \mathbf{S}')\Big)}_{\text{repayment}} \mid \mathbf{S} \Big] \end{aligned} \tag{12}$$

In equations (11)-(12), $B'' = B''(B', B'_u, \mathbf{S}')$ and $B''_u = B''_u(B', B'_u, \mathbf{S}')$ are the equilibrium policy functions of the government, while variables q^d and q^d_u denote the prices of defaulted bonds. These prices are positive because the lenders can expect to recover a fraction ω of face value upon a resolution of default. The defaulted bond prices are given by

$$\begin{aligned} q^d(B, B_u, \mathbf{S}) = \mathbb{E}\Big[& (1-\theta) \underbrace{q^d\Big((1+r)B, (1+r)B_u, \mathbf{S}'\Big)}_{\text{continued exclusion}} \\ & + \theta\omega \underbrace{\Big[1 - d\Big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\Big)\Big]\Big[1 + (1-\delta) q(B', B'_u, \mathbf{S}')\Big]}_{\text{readmission and repayment}} \\ & + \theta\omega \underbrace{d\Big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\Big) \, q^d\Big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\Big)}_{\text{readmission and default}} \mid \mathbf{S} \Big] \end{aligned} \tag{13}$$

for the regular maturity debt and

$$q_u^d(B, B_u, \mathbf{S}) = \mathbb{E}\Big[(1-\theta)\ \underbrace{q^d\big((1+r)B, (1+r)B_u, \mathbf{S}'\big)}_{\text{continued exclusion}} \\ +\theta\omega\ \underbrace{\Big[1 - d\big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\big)\Big]\Big(\kappa + q_u(B', B_u', \mathbf{S}')\Big)}_{\text{readmission and repayment}} \\ +\theta\omega\ \underbrace{d\Big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\Big)\ q_u^d\Big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\Big)}_{\text{readmission and default}}\ |\ \mathbf{S}\Big] \tag{14}$$

for the ultralong debt. In equations (13)-(14), $B' = B'\Big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\Big)$ and $B_u' = B_u'\Big(\omega(1+r)B, \omega(1+r)B_u, \mathbf{S}'\Big)$ are the equilibrium policy functions of the government. Every period following a default, the government has a probability θ of being readmitted to the markets. If that happens, both debt stocks undergo a uniform haircut of $1-\omega$ and the government can again decide whether to repay or default. While in default, the government's liabilities towards foreign lenders grow at the current risk-free rate.

Concluding this section, Definition 1 introduces the concept of Markov Perfect Equilibrium, which is often applied in the analysis of quantitative models of sovereign debt.

Definition 1 *A Markov Perfect Equilibrium consists of the government value functions* $v(B, B_u, \mathbf{S})$, $v^r(B, B_u, \mathbf{S})$, $v^d(B, B_u, \mathbf{S})$; *policy functions* $C(B, B_u, \mathbf{S})$, $B'(B, B_u, \mathbf{S})$, $B_u'(B, B_u, \mathbf{S})$, $d(B, B_u, \mathbf{S})$; *and bond price schedules* $q\big(B', B_u', \mathbf{S}\big)$, $q_u\big(B', B_u', \mathbf{S}\big)$, $q^d\big(B, B_u, \mathbf{S}\big)$, $q_u^d\big(B, B_u, \mathbf{S}\big)$ *such that:*

1. *Policy function d solves the government's default-repayment problem.*
2. *Policy functions $\{C, B', B_u'\}$ solve the government's consumption-saving problem.*
3. *Bond price functions $\{q, q_u, q^d, q_u^d\}$ are such that international lenders make zero profit.*

3.4 Optimal maturity structure

In this section, I analyze the forces underlying the government's optimal choice of the maturity structure. Similar to Arellano and Ramanarayanan (2012), this decision will arise as a solution to the tradeoff between the incentive benefit of regular debt stock, and the hedging benefit provided by the ultralong debt.

Suppose that the bond price functions and the value of repayment are all continuous and differentiable. The first-order necessary conditions for an interior solution are then

$$u'(C)\Big[q + \frac{\partial q}{\partial B'}\big(B' - (1-\delta)B\big) + \frac{\partial q_u}{\partial B'}\big(B'_u - B_u\big)\Big] = \beta\mathbb{E}\Big[u'(C')\big(1 + (1-\delta)q'\big) \mid \mathbf{S}, d' = 0\Big]$$
$$u'(C)\Big[q_u + \frac{\partial q}{\partial B'_u}\big(B' - (1-\delta)B\big) + \frac{\partial q_u}{\partial B'_u}\big(B'_u - B_u\big)\Big] = \beta\mathbb{E}\Big[u'(C')\big(\kappa + q'_u\big) \mid \mathbf{S}, d' = 0\Big]$$

where $d' = d'(B', B'_u, \mathbf{S}')$ is the government's optimal decision rule for default (i.e. the expectations are conditional on repayment). The left-hand side of each of these conditions represents the marginal benefit of issuing an extra unit of the respective bond, measured in the utility from consumption. Importantly, the government knows that by issuing more debt, it exposes itself to a higher risk of default, affecting negatively the prices of all outstanding bonds. The right-hand side represents the marginal cost of issuing an extra unit of debt. In all states where the government finds it optimal to repay its debt, it must pay the coupon and repurchase the portion of the bond that does not mature. We can re-write the equations above and use the definition of bond prices (11)-(12) to obtain

$$u'(C)\underbrace{\Big[1 + \frac{\partial q}{\partial B'}\frac{B' - (1-\delta)B}{q} + \frac{\partial q_u}{\partial B'}\frac{B'_u - B_u}{q}\Big]}_{\text{Incentive term}}$$
$$= \beta(1+r)\mathbb{E}\Big[u'(C') \mid \mathbf{S}, d' = 0\Big]\underbrace{\frac{\mathbb{E}\Big[u'(C')\big(1 + (1-\delta)q'\big) \mid \mathbf{S}, d' = 0\Big]}{\mathbb{E}\Big[u'(C') \mid \mathbf{S}, d' = 0\Big]\mathbb{E}\Big[1 + (1-\delta)q' \mid \mathbf{S}, d' = 0\Big]}}_{\text{Hedging term}} \tag{15}$$

$$u'(C)\underbrace{\Big[1 + \frac{\partial q}{\partial B'_u}\frac{B' - (1-\delta)B}{q_u} + \frac{\partial q_u}{\partial B'_u}\frac{B'_u - B_u}{q_u}\Big]}_{\text{Incentive term}}$$
$$= \beta(1+r)\mathbb{E}\Big[u'(C') \mid \mathbf{S}, d' = 0\Big]\underbrace{\frac{\mathbb{E}\Big[u'(C')\big(\kappa + q'_u\big) \mid \mathbf{S}, d' = 0\Big]}{\mathbb{E}\Big[u'(C') \mid \mathbf{S}, d' = 0\Big]\mathbb{E}\Big[\kappa + q'_u \mid \mathbf{S}, d' = 0\Big]}}_{\text{Hedging term}} \tag{16}$$

Equations (15) and (16) reveal how the optimal maturity structure depends on the behavior of bond prices in equilibrium. The left-hand side of each formula contains an *incentive term* that affects the marginal revenue from issuing debt. This term arises due to the fact that higher debt increases future likelihood of default and depends on the relative elasticities of bond prices with respect to issuing an extra unit of a bond. The incentive term is typically lower for longer-term debt because such debt makes the future government less concerned about keeping bond prices high and leads to higher dilution. As a result, bond prices decrease faster today. The right-hand side of each formula contains a *hedging term* which is related to the comovement between tomorrow's marginal utility of consumption and the bond prices. The key insight in this paper is that, apart from the standard channel explained by Arellano and Ramanarayanan (2012), additional hedging motive arises in the presence of shocks to the risk-free interest rate. This is because, while such shocks go hand-in-hand with the marginal utility of an impatient borrower, they are also mechanically in an inverse relationship with bond prices. Hence, the numerator of each hedging term is smaller than the denominator, reducing the marginal cost of issuing debt.

What, then, motivates the issuance of ultralong sovereign bonds as opposed to long-term debt with average maturity that also provides hedging benefits? The answer is quantitative and lies in the nature of the risk-free rate uncertainty. Taking the ratio of the right-hand side of equations (16) and (15), we obtain the relative hedging benefit of the ultralong debt

$$\text{relative hedging benefit} = \frac{\dfrac{\mathbb{E}\left[u'(C')\left(\kappa+q_u'\right) \mid \mathbf{S}, d'=0\right]}{\mathbb{E}\left[u'(C') \mid \mathbf{S}, d'=0\right]\mathbb{E}\left[\kappa+q_u' \mid \mathbf{S}, d'=0\right]}}{\dfrac{\mathbb{E}\left[u'(C')\left(1+(1-\delta)q'\right) \mid \mathbf{S}, d'=0\right]}{\mathbb{E}\left[u'(C') \mid \mathbf{S}, d'=0\right]\mathbb{E}\left[1+(1-\delta)q' \mid \mathbf{S}, d'=0\right]}} \tag{17}$$

Given that risk-free interest rate is highly persistent in the data, while emerging market's sovereign debt tends to have short maturities, the covariance between marginal utility and the prices of regular long-term debt is small (in absolute terms). On the other hand, prices of the ultralong debt incorporate an expectation of a much longer path of future rates and thus covary much more with marginal utility as a result of the shocks to risk-free rate. On the other hand, ultralong debt provides many fewer incentives for the government to be concerned about future bond prices. Taking the ratio of the left-hand side of equations (16) and (15), we then obtain the relative incentive benefit of the regular debt

$$\text{relative incentive benefit} = \frac{1 + \frac{\partial q}{\partial B_u'}\frac{B'-(1-\delta)B}{q_u} + \frac{\partial q_u}{\partial B_u'}\frac{B_u'-B_u}{q_u}}{1 + \frac{\partial q}{\partial B'}\frac{B'-(1-\delta)B}{q} + \frac{\partial q_u}{\partial B'}\frac{B_u'-B_u}{q}} \tag{18}$$

The optimal maturity structure arises from borrower setting the relative hedging benefit equal to the relative incentive benefit. The question of how much of the ultralong debt we can observe in equilibrium, and in what states it is most likely to arise, is quantitative in nature. Section 4 seeks to answer these questions by taking the model to the data.

4 Quantitative analysis

In this section I calibrate the model by specifying the functional forms, choosing parameter values and discussing the numerical approach. I then illustrate the mechanism of the model by analyzing the strength of the relative forces at play and plotting policy functions and bond price schedules.

4.1 Solving the model

The representative household's utility is a CRRA function of the form $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$. The model is solved numerically by value function iteration using a continuous choice of next period debt and a two-dimensional tensor-product cubic spline interpolation to evaluate off-grid points, similarly as described in Hatchondo, Martinez and Sapriza (2010). I employ the Powell algorithm to solve the debt portfolio problem in two dimensions. I use 41 points for the grid of regular debt, and 21 points for the grids of ultralong debt and growth, as well as 9 grid points for the risk-free rate. Expectations are approximated using Gaussian quadrature with 21 nodes for the growth shock, and 9 nodes for the risk-free rate shock. Linear interpolation is used to evaluate equilibrium objects at off-grid nodes.

To facilitate the model's convergence, I introduce a utility cost of adjusting the amount of ultralong debt of the form $\frac{\psi}{2}\left(B'_u - B_u\right)^2$. While theoretically the maturity choice can be pinned down in this model, in practice there are many states in which the government is nearly indifferent between many potential portfolios, and any mistaken choices feed back into the bond prices further propagating the error. Adding a utility cost of adjusting the ultralong debt resolves this indifference and can be justified empirically by the fact that sovereign century bonds are rare securities in the financial markets. Any new issuance or buyback of such bonds would require an amount of effort from the government beyond what it exerts for usual fiscal policy operations. In practice, I find that in the simulated equilibrium of the model the maximum utility cost incurred due to adjusting the stock of ultralong bonds is equivalent to 0.04% of current period consumption.

In models with long-term debt and positive recovery rate the government finds it optimal in some states to issue the maximum amount of debt (because bond prices do not fall below the price of defaulted bonds). To eliminate this behavior, I add a standard constraint that positive net issuances must occur at price above a certain minimum (set separately for the two types of debt). I find that this constraint is never binding in simulations.

4.2 Data

The data used in this section comes from several different sources. Mexico's national accounts data comes from INEGI (1995) and the World Development Indicators. External debt series is acquired from WDI and covers the period of 1970-2016. Risk-free interest rate series (short and long) are taken from OECD, together with the 2012 long-term baseline projections discussed in Section 4.2. All bond yield data are downloaded from Bloomberg and spreads are calculated as differences between the yields on Mexican bonds and the corresponding risk-free yields.

4.3 Calibration

The identification strategy for the model's parameters is in line with the general approach in the literature. In what follows, I first calibrate the parameters of the stochastic processes for endowment growth and risk-free interest rate separately from the core model. Then, I pick the remaining structural parameters in part from the literature and partly to match certain very general characteristics of Mexico's borrowing behavior, such as average amount of debt and the frequency of defaults.

Given these targets, the model is then evaluated by its ability to match the facts about the behavior its endogenous objects such as debt maturity structure and bond spreads.

4.3.1 Estimating the Mexican growth process

Table 2 summarizes the parameters of Mexico's GDP growth rate for years 1980-2016. This time period is selected because of the apparent regime change in the data around 1980, evident in Figure 3. As mentioned earlier, although Aguiar and Gopinath (2006) proposed a model with some persistence in trend growth, I assume it away for two reasons. First, the most recent GDP growth data for Mexico shows little support for it, yielding estimates of the persistence coefficient that are quantitatively small and statistically insignificant. Second, dispensing with persistent growth allows me to simplify the model and focus on the novel source of uncertainty, i.e. the shocks to risk-free interest rate.

Table 2: Parameters of Mexico's growth rate process

Parameter	Coeff.	St. Error	P-value	RMSE
μ	0.026	0.006	0.000	0.033

4.3.2 Estimating the risk-free interest rate process

I now estimate the parameters of the law of motion for risk-free interest rate. To do so, I proceed in two steps. First, I fit an AR(1) process for the 10-year real yield[9] by combining the historical data up to 2012 and the long-term projection up to 2060, as presented in Section 2.3. Minimizing the sum of squared deviations between OECD forecasts for years 2012-2060 and the analogous model-generated forecasts, I find the unconditional mean of the 10-year rate to be 4.22% with a persistence of 0.86. Then, I use the data on 10- to 1-year term premium for US bonds in years 1954-2012 and infer the unconditional mean of the 1-year interest rate to be 3.1%. Finally, I estimate an AR(1) process given in equation (3) for the 1-year rate using historical data under the aforementioned restriction on unconditional mean. Table 3 summarizes the parameter values obtained in this way. The calibration technique that uses forecasts, in addition to historical data, follows the approach of Paluszynski (2023) who points out that accounting for real time GDP forecasts can help the standard model better explain the European debt crisis of 2009-2012. In contrast to that paper, here I use long-term projections that span the period of 50 years.[10]

Table 3: Parameters of the risk-free interest rate process

Parameter	$\bar{r}$	ρ_r	σ_r
Value	0.031	0.818	0.019

[9] The real long-term yields are obtained with the Fisher equation using the current CPI inflation rate. This implicitly assumes that inflation rate follows a random walk process, which need not be the case in reality. To ensure robustness, I check the 10-year bonds yields on the inflation-indexed TIPS bonds available since 1999 and verify that they are quantitatively similar.

[10] A usual concern involves accuracy and representativeness of OECD forecasts. While the consistent sources of long-horizon projections are scarce, in Paluszynski (2023) I compare the forecasts published by international organizations such as the OECD, IMF or European Commission, against the Consensus Economics forecasts, which aggregate the beliefs of private institutions. I show that the former not only are not much worse, but often outperform the markets' predictions.

It is natural to ask how these estimates compare to ones that we would obtain by using historical data only. Estimating equation (3) using the 1965-2016 data only, we get $\bar{r} = 0.013$, $\rho_r = 0.74$ and $\sigma_r = 0.019$. Hence, apart from the unconditional mean, the parameter values are similar. Section 5.5 shows that the results under these two sets of parameter values are generally similar, although using the OECD forecast elevates the amount of risk coming from interest rate fluctuations and leads the government to issue more ultralong debt on average.

4.3.3 Calibrating the structural parameters

Table 4 summarizes the calibration of structural parameters in the model. The first group of parameters is selected using outside information and standard values from the literature. The risk aversion γ is 2, which is a typical value in macroeconomic models. The probability of re-entry following a default θ is assumed to be 0.33, a value in the middle of the range typically used by sovereign default models. The recovery rate ω is difficult to calibrate as the literature has reported a wide interval of possible values. I pick 0.52 which is the haircut applied following the 2012 Greek default, the most notable default episode in recent past. This also happens to be somewhat of a median value among the ones reported in other studies. The coupon rate κ is taken directly from Mexico's 100-year bond denominated in USD (as presented in Table 1). The regular debt portfolio maturity profile parameter δ is assumed to be 0.2 resulting in expected term to maturity of 5 years,

Table 4: Calibration of the structural parameters

Symbol	Meaning	Value	Source
γ	Risk aversion	2	Standard
θ	Re-entry probability	0.33	Standard
ω	Recovery rate	0.48	Greek haircut
δ	Probability of maturing	0.2	Mexican debt data
κ	Ultra coupon rate	5.75	
ϕ	Default cost	0.055	Joint calibration
β	Discount factor	0.813	
ψ	Adjustment cost	100.00	Heuristic[11]
Calibration targets		**Model**	**Data**
E (debt/GDP)		35.0	35.0
Default probability		3.00	3.00

Note: targeted moments are given in percentage points.

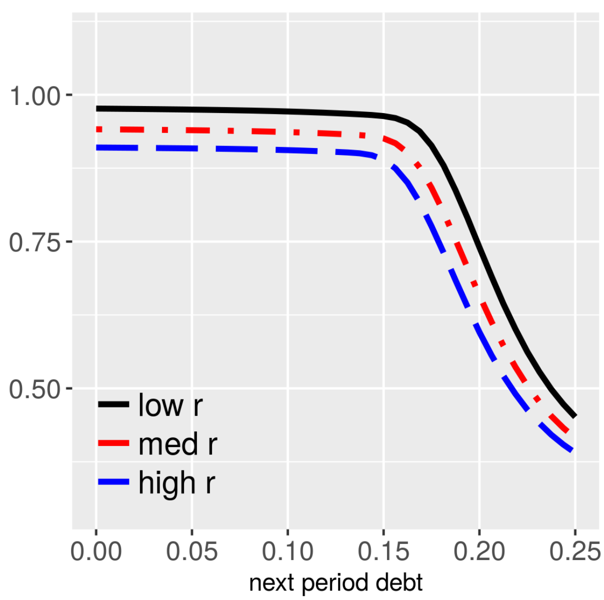
[11] I select the smallest value of ψ needed to achieve robust convergence of the model.

roughly in line with the average length of Mexico's foreign bonds. The ultralong debt adjustment cost parameter ψ is set to a minimum value that ensures convergence of the model, 100 in this case. The remaining two parameters ϕ and β are selected jointly in the Simulated Method of Moments procedure to match two general target statistics: mean external debt-to-GDP[12] in years 1980-2016 of 35.0%, and the ergodic default frequency of 3%. As the lower panel of Table 4 shows, the model matches these two targets exactly.[13]

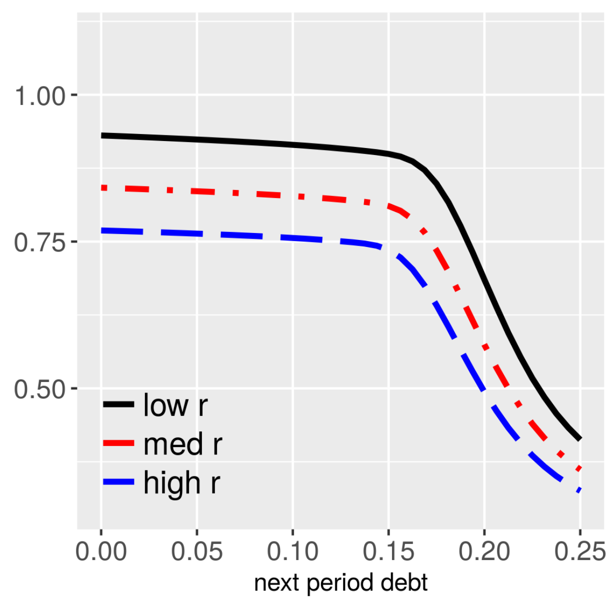
4.4 Bond prices and policy functions

In this section I visualize some properties of the equilibrium by first looking at the bond price schedules, and then analyzing the policy functions for debt accumulation.

Figure 5 depicts the bond price schedules as function of next period debt B', and for three separate realizations of the risk-free rate.[14] As it is common for this class of models, bond



(a) Shorter maturity



(b) Ultralong

Figure 5: Bond prices as function of next period debt for different values of risk-free rate

[12]In the model, debt is defined as the expected present value of all future debt liability cash flows.

[13]For practical purposes, I calibrate the model in which ultralong debt and the maturity choice are shut down. This allows me to speed up the computation significantly. As will become clear, the government chooses rather small amounts of the ultralong bonds in equilibrium so that the total average amount of debt and the unconditional default probability do not deviate significantly from targets when the optimal maturity choice is turned on.

[14]For illustration, the coupon rates are rescaled in Figure 5 such that the expected payoff next period

prices are mostly flat for small amounts of future debt, and then at some point they drop sharply eventually converging to the prices of defaulted bonds as the default probability nears 1. Panel 5(a) shows that while the prices of regular maturity debt do vary for different risk-free rates, the differences between them are rather small. On the other hand, the ultralong debt prices differ much more, where a low risk-free rate results in a high q_u and vice-versa. This covariance between r and q_u is a crucial factor behind the hedging mechanism of the ultralong debt described in Section 3.4. Figure 5 shows that while the mechanism is naturally present for both types of long-term debt, its quantitative power is potentially more significant for the ultralong debt. Section 5.4 further formalizes this point. On the other hand, notice that the prices of ultralong bonds are also lower than the prices of regular bonds which is caused by the fact that longer-term debt promotes more debt dilution.

Figure 6 presents policy rules for the accumulation of debt as function of current-period regular debt, and for different values of the risk-free interest rate. Panel 6(a) shows that as the amount of outstanding debt increases, the government's issuance of the regular debt also goes up. In addition, a lower risk-free rate leads to higher issuance for all states today. Panel 6(b) shows the decision rules for issuing the ultralong debt (assuming that the amount currently outstanding is zero). Notice that for the lowest values of B, the government does not decide to issue any ultralong bonds. For intermediate and high levels of outstanding debt, the government issues small amounts of the perpetuity when the risk-free rate is low. At medium values of the risk-free rate, the government only issues an ultralong bond for very high levels of current debt. On the other hand, when current risk-free rate is high, no issuances of the ultralong debt are observed.

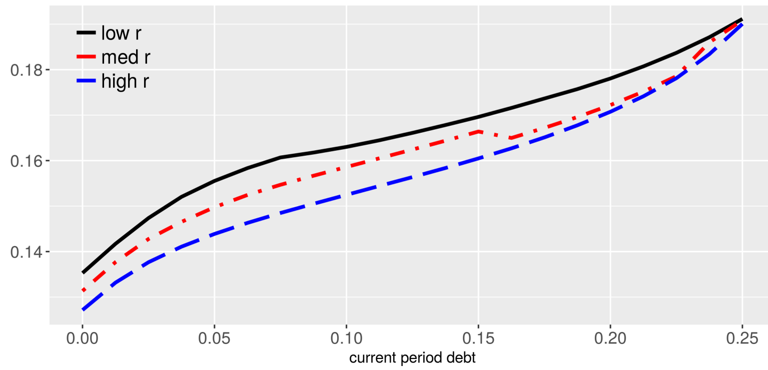
5 Results

This section presents the main simulation results and discusses how the predicted maturity choices and bond spread statistics compare to the ones observed in the data.

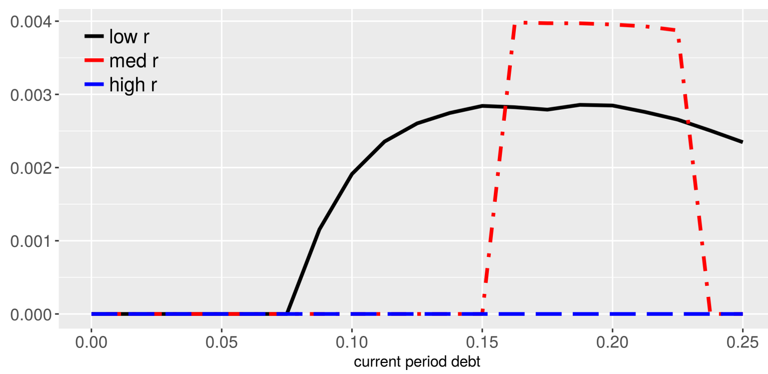
5.1 Business cycle moments

I start the analysis by evaluating some typical non-targeted business cycle statistics, presented in Table 5. As it is common for this type of framework, the model generates highly volatile consumption and trade balance, although it does not achieve the full variance of

from both bonds is normalized to 1.



(a) Shorter maturity



(b) Ultralong

Figure 6: Policy functions for debt accumulation for different values of risk-free rate

Mexican data covering the period from 1980 to 2016. The measurement of these moments is very sensitive to the assumed time period. For example, starting the sample in mid 1990s results in consumption volatility that is similar to that of income growth. Hence, the moments produced by the model appear to be a reasonable fit. It is also worth noting that trade balance in the model is strongly countercyclical, an expected feature in the models of emerging markets and serial defaulters.

Table 5: Non-targeted business cycle moments

	Model	**Data**
SD(c)/SD(g)	1.32	1.52
SD(tby)/SD(g)	0.40	0.85
corr(c,g)	0.98	0.62
corr(tby,g)	−0.74	−0.4

Note: data covers the period 1980-2016. Consumption is normalized by the previous period GDP, like in the model.

5.2 What generates defaults?

To understand the behavior of new elements in the model, it is instructive to consider the determinants of default. Figure 7 presents a scatter plot of the binary outcomes of government's repayment-default problem in the simulated ergodic distribution as function of growth shocks and changes in interest rate.[15] The graph shows that defaults generally tend to occur when growth rates in the economy are low. Importantly, while defaults occur at all interest rate levels, the default threshold increases with the change in the risk-free rate. In particular, when interest rate falls sharply the economy defaults at around -5% growth rate, while when it increases a default can occur with -2.5% growth.

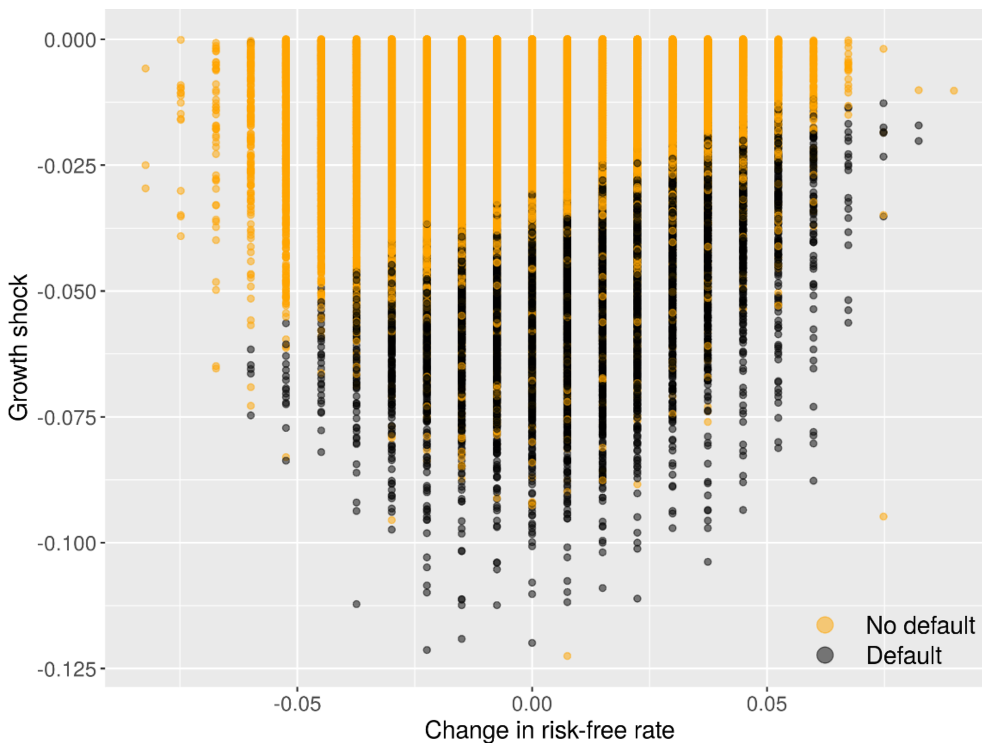
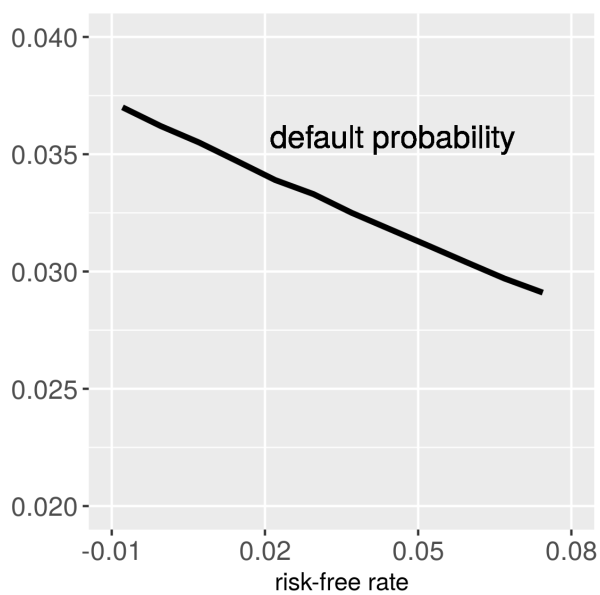


Figure 7: Growth shocks vs. interest rate shocks in the simulated model

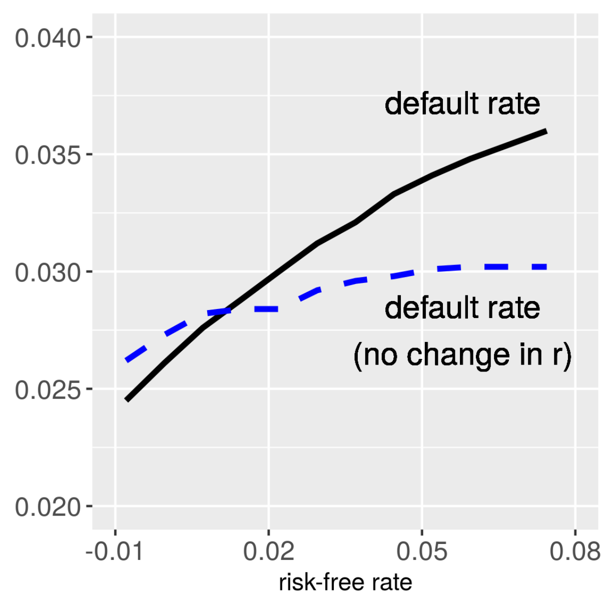
Next, it is interesting to see how default depends on the current interest rate level. Figure 8 presents this information in two forms - the expected default probability in next period, as well as the realized default frequency, each averaged at different levels of risk-free rate. As can be noticed, the former is a decreasing function of the risk-free rate which implies

[15]The ergodic distribution is obtained by averaging over 1800 simulated paths, each consisting of 10,000 periods. In each path, 5 periods are burned following a default. For practical reasons, to generate the graph in Figure 7, I restrict the sample to 50 paths chosen at random. The scatter plot takes the form of vertical lines because interest rate is kept on a discrete grid in the simulations.

that lower interest rate is associated with higher default risk. However, the latter *increases* with r which means that low rate states are on average marked with fewer defaults. What generates these opposing patterns is the fact that, when interest rate is low it is bound to increase by mean reversion, exposing the government to additional default risk as shown in Figure 7. By contrast, high interest rate today implies that it is likely to go down tomorrow making default less probable. To appreciate this point better, the right-hand side panel of Figure 8 additionally plots the realized default frequency conditional on there being no change in the risk-free rate from the previous period. We can see that this conditional default frequency is much flatter as a function of r which means that default risk is roughly uniform in the absence of interest rate shocks.



(a) Expected default probability



(b) Realized default frequency

Figure 8: Expected vs. realized default probability as function of risk-free rate

5.3 Maturity choice and spreads in ergodic distribution

I now turn to the problem of maturity choices and spread dynamics over the risk-free rate cycle in the model. Table 6 presents the average duration and share of ultralong bonds in total debt[16], as well as their correlations with the two stochastic state variables. The model predicts an average share of ultralong bonds in total debt of 5% which is remarkably close

[16]The share of ultralong debt is defined as $\frac{q_u^{rf} B_u'}{q_u^{rf} B_u' + q^{rf} B'}$, where q^{rf} and q_u^{rf} are the prices of risk-free bonds with regular and ultralong maturity, respectively. Duration of debt is the Macaulay duration defined as

to the share that Mexico actually issued in years 2010-2015. The share of ultralong bonds in total *new issuances* of debt is smaller than 1% but has a large standard deviation of 3.27%. The lower panel of Table 6 reveals that these shares are strongly and negatively correlated with the risk-free interest rate. The reason behind this relationship lies in the comovement of risk-free interest with the determinants of optimal maturity choice defined in equations (17)-(18). As interest rate declines, so does the relative hedging term which makes ultralong debt more attractive in such states due to the enhanced insurance it provides. At the same time, the relative incentive term increases weakly which makes such issuances more feasible. Figure 9(a) plots average shares of the outstanding and newly issued ultralong debt at different levels of the risk-free rate. Notice that for the levels of r above 5%, the share of the outstanding ultralong debt drops below 3%, while the government actually buys these bonds back.

The spread curve is roughly flat on average in the model which is expected given that lenders are risk-neutral. Interestingly, as Figure 9(b) shows, the spread curve slopes *downwards* when interest rate is low and *upwards* only when it takes average and high values. This result is closely related to the features demonstrated by Figures 8(a) and 8(b), i.e. states with low interest rates are more risky than states with higher rates. The lower panel of Table 6 confirms that this additional risk comes from the scope of future interest rate changes. While the expected default probability is negatively correlated with r, the realized default frequency covaries positively with it, and the relationship is weak.

5.4 Quantifying the mechanism

Why does ultralong debt provide a hedging opportunity for the borrower, and why does it become stronger when the risk-free rate falls? To understand the underlying mechanism better, I now quantify the individual factors that drive maturity choices in the model. The upper panel of Table 7 shows that prices of ultralong bonds are much more volatile relative to their mean than prices of bonds with regular maturity. This is because, as illustrated in Figure 5, prices of debt with longer maturity are affected much more by shocks to a persistent risk-free rate. Consequently, also the overall value of the ultralong debt has a much higher coefficient of variation than the value of regular debt and it provides a better hedge against such shocks. The resulting *relative hedging* term as defined in (17)

$\frac{\sum_{t=1}^{\infty}\left\{B\frac{t(1-\delta)^{t-1}}{(1+y)^t}+B_u\frac{t\kappa}{(1+y_u)^t}\right\}}{\sum_{t=1}^{\infty}\left\{B\frac{(1-\delta)^{t-1}}{(1+y)^t}+B_u\frac{\kappa}{(1+y_u)^t}\right\}}$, where $y=\frac{1}{q}-\delta$ is the yield to maturity on regular bonds, while $y_u=\frac{\kappa}{q_u}$ is the yield to maturity on ultralong bonds.

Table 6: Ergodic moments of maturity choices and spreads

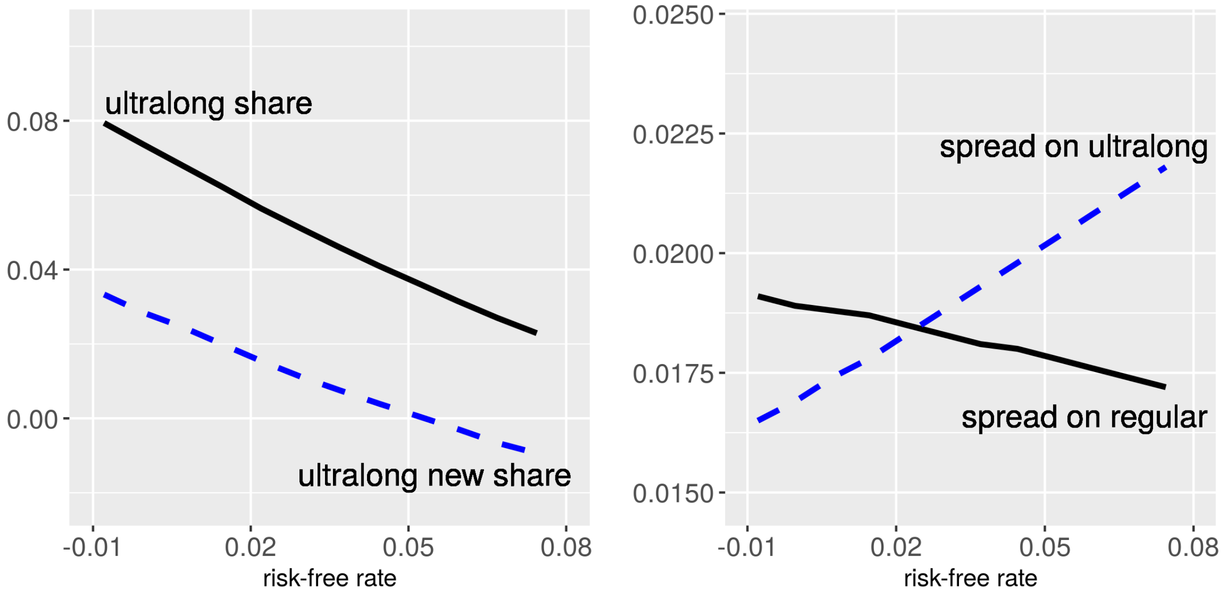
	Mean	**St. dev.**
ultra share	5.03	2.39
ultra new share	0.76	3.27
duration	5.12	0.91
spread reg.	1.86	0.14
spread ultra	1.96	0.25
spread crve reg/ultra	0.04	0.33
Correlation with:	r	g
ultra share	−0.96	−0.05
ultra new share	−0.52	−0.05
duration	−0.98	−0.01
spread reg	−0.52	−0.74
spread ultra	0.99	−0.07
spread crve reg/ultra	0.95	0.27
debt/y	−0.99	−0.05
tb/y	0.61	−0.74
default prob.	−0.36	−0.81
default frequency	0.03	−0.38
rel. incentive	−0.33	−0.03
rel. hedging	0.83	−0.17
u'(c)	0.18	−0.98

Note: except for duration, the moments in the top panel are given in percentage points.

naturally takes values smaller than one.

The lower panel of Table 7 calculates various measures conditional on current risk-free rate being strictly below or above its unconditional mean. Because low interest rate states are inherently more risky than high interest rate ones, as Figure 8(a) showed, the resulting volatility of next period marginal utility of consumption and prices of either both bond types are higher in these states. Consequently, next period bond prices covary more strongly with the marginal utility when the risk-free rate is low than when it is high. This produces a lower average value of the relative hedging term in these states, resulting in higher average share of the ultralong bonds in total debt.

Figures 10(a) and 10(b) explain in more detail how these considerations lead the government to choose more ultralong bonds when the risk-free rate is low. In this exercise, I select a state with a typical level of income and current debt stocks. Holding the opti-



(a) Shares of ultralong debt

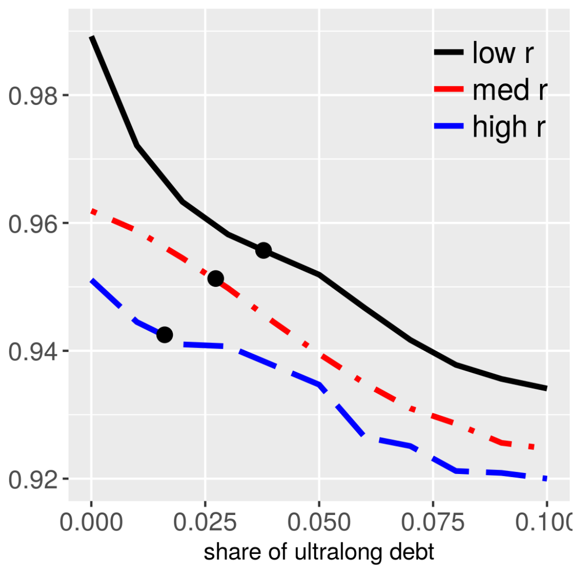
(b) Spreads on two debt types

Figure 9: Maturity choices and bond spreads as function of the risk-free rate

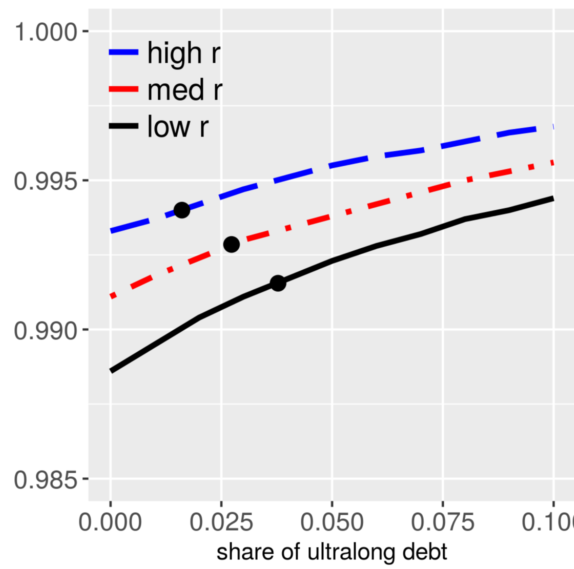
Table 7: Drivers of the maturity composition

	Mean	**St. dev.**
q	3.96	0.38
q_u	2.62	0.47
$\big[1+(1-\delta)q\big]b$	0.33	0.03
$\big[\kappa+q_u\big]b_u$	0.01	0.01
Interest rate:	**below mean**	**above mean**
SD $u'(c')$	0.081	0.079
SD q'	0.48	0.37
SD q'_u	0.46	0.32
Corr$\big(u'(c'), q'\big)$	-0.38	-0.33
Corr$\big(u'(c'), q'_u\big)$	-0.33	-0.27
E (rel. hedging)	0.994	0.995
E (rel. incentive)	1.010	0.993
E (ultra share)	6.84	2.48
E (ultra new share)	2.61	-0.74

mal level of *total* debt fixed, I counterfactually vary the share of ultralong bonds that the government could choose. I do so for three current interest rate states - low, medium and high - and measure the relative incentive term, as in (18), and the relative hedging term, as in (17). The graphs show that a low current risk-free rate is associated with stronger incentives to issue ultralong bonds, as well as a higher extent of hedging associated with it. The resulting optimal choice in equilibrium implies that the government issues more ultralong debt when interest rate is low, and reduces its share when interest rate becomes high (marked with black dots).



(a) Relative incentives

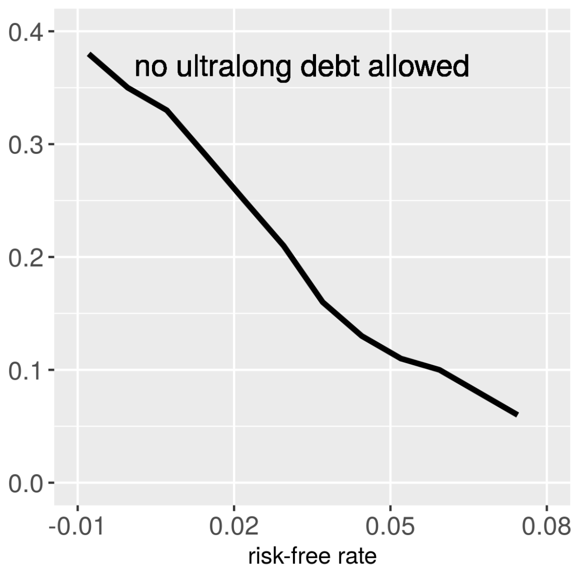


(b) Relative hedging

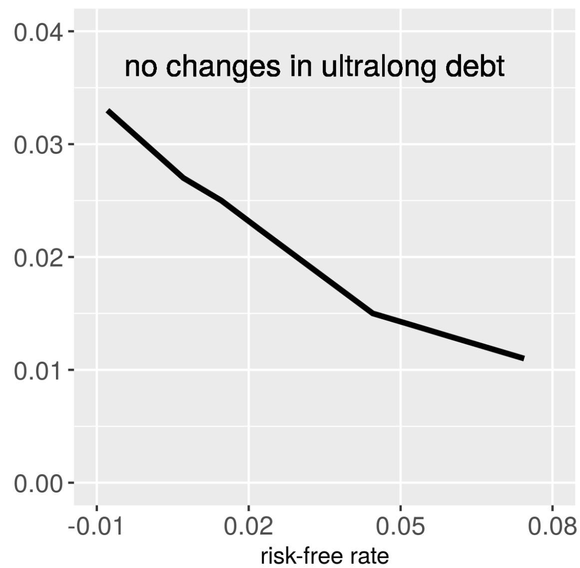
Note: holding the total debt level fixed, I measure the relative incentive and hedging terms, as defined in formulas (18)-(17). Black dots denote the optimal choice of the ultralong share in equilibrium for this particular state.

Figure 10: Relative incentives and relative hedging for counterfactual debt portfolios

Finally, it is interesting to ask about the welfare implications of the hedging provided by ultralong bonds in the model. To provide an answer, I measure a consumption-equivalent compensation that the government would require in order to give up its optimal level of the ultralong debt. I assume that the total debt level is fixed at the same level as in equilibrium and consider two scenarios: i) no ultralong debt is allowed; and ii) ultralong debt is kept unchanged (in undetrended terms). Figure 11 plots these compensation measures averaged for different levels of interest rate in the simulations. Panel 11(a) shows that the compensation required to forgo issuing ultralong bonds is equivalent to 0.4% of current consumption when the risk-free rate is equal to -1% and declines to less than 1% as the rate approaches 8%. These numbers appear large due to the fact that the government is often forced to buy back the outstanding ultralong bonds which increases the price of regular bonds as well. Hence, panel 11(b) plots the compensation required to keep the current undetrended stock of the ultralong debt unchanged. Such a compensation varies between 0.01% to 0.04% of current consumption. Crucially, both panels of Figure 11 confirm that the hedging benefits provided by the ultralong bonds are most valuable when interest rates are low.



(a) No ultralong debt allowed



(b) No changes in ultralong debt allowed

Note: compensation is measured in percentage of current consumption, averaged across different levels of interest rate.

Figure 11: Consumption-equivalent compensation related to restricted policies

5.5 Comparison with benchmarks models

I now show that a time-varying and persistent risk-free interest rate is an essential ingredient to generate relevant predictions for the choice of ultralong debt. To do so, I contrast the results above against two restricted variants of the model, one with a constant risk-free rate and one where it is *i.i.d*. Table 8 presents a collection of relevant moments describing the behavior of these two models. It also shows the behavior of two models with a persistent risk-free rate: one where the stochastic process is estimated using historical data only and no forecasts, as well as the (preferred) model in which the interest rate process is estimated using both the historical data and long-term forecasts, as described by Table 3.[17]

[17]Concretely, in the *i.i.d.* rate model, I assume that $\rho_r = 0$, while in the constant rate model I assume that $\rho_r = \sigma_r = 0$. The no-forecast model assumes that $\bar{r} = 1.27\%$, $\rho_r = 0.74$ and $\sigma_r = 1.94\%$. The forecast model

For simplicity, I do not recalibrate these alternative models because the targeted moments are close to the data targets. It should first be noticed that the models with constant and *i.i.d* rates produce very similar business cycle statistics. This suggests that *i.i.d.* shocks to the risk-free rate do not pose a major problem for the government who is mostly able to insure against them. Importantly, in both models the government chooses not to issue any ultralong debt. This is because the relative hedging term, as defined in (17) is essentially equal to one implying that ultralong debt does not provide any additional scope for hedging, beyond what the regular maturity debt does. By contrast, in the model where interest rate is persistent the relative hedging term is strictly smaller than one, resulting in the government holding around 5% of its total debt in ultralong bonds. The (preferred) model in which the risk-free rate process is estimated using the OECD forecast produces a higher average share due to the fact that r has an overall higher volatility under these parameters. The difference is not large though, showing that the estimation method is not crucial for the theory.

Table 8: Comparison of models with different assumptions about the risk-free rate

	constant rate	***i.i.d.* rate**	**persistent rate**	
			no forecast	**with forecast**
debt/GDP	34.2	34.2	35.5	36.2
default probability	2.91	2.93	3.03	3.13
ultra share	0.00	0.00	4.87	5.17
duration	4.11	4.11	6.19	5.15
spread reg.	1.99	2.00	1.87	1.85
spread ultra	1.97	1.99	1.88	1.95
SD(c)/SD(g)	1.29	1.29	1.30	1.32
SD(tby)/SD(g)	0.29	0.30	0.34	0.40
rel. hedging	1.00	1.00	0.998	0.998

Note: the models with constant and i.d.d. are based on the same structural parameters as the main model. The persistent rate - no forecast model refers to the model in which the stochastic process for the risk-free rate is estimated using historical data only. The calibrated parameters for that model are $\beta = 0.834$ and $\phi = 0.048$.

6 Risk Premia in Sovereign Bonds

To further analyze the spreads on ultralong bonds, I augment the main model by assuming that foreign lenders are risk averse. Investor risk aversion has been shown to be an

is the main calibrated model as shown in Table 3.

important part of sovereign spreads, for example by Borri and Verdelhan (2011), Longstaff et al. (2011) and more recently Tourre (2017). Borrowing from the finance literature, I introduce risk-averse lenders by directly modeling their stochastic discount factor.

6.1 Pricing kernel

In this section, I develop and estimate a two-factor affine term structure model. Formally, the lenders evaluate a stochastic stream of debt payments $\{\ell_t\}_{t=0}^{\infty}$ using a pricing kernel $M_{t,t+1} = exp\{m_{t,t+1}\}$, i.e. they evaluate $\mathbb{E}_0 \sum_{t=0}^{\infty} \Pi_{s=0}^{t} M_{s-1,s}\ell_t$. The stochastic discount factor is modeled using a simplified specification of Lettau and Wachter (2011) and takes the form

$$m_{t,t+1} = -r_t - \frac{1}{2}a_t^2\sigma_x^2 - a_t\sigma_x\varepsilon_{x,t+1} \tag{19}$$

$$x_{t+1} = (1-\rho_x)\mu_x + \rho_x x_t + \varepsilon_{x,t+1} \tag{20}$$

$$r_{t+1} = (1-\rho_r)\bar{r} + \rho_r r_t + \varepsilon_{r,t+1} \tag{21}$$

$$a_t = \alpha_0 + \alpha_1 x_t \tag{22}$$

The SDF depends on two underlying factors, the current level of the risk-free rate r_t, as well as the "foreign fundamentals" factor x_t which affects it through the price of risk variable a_t and the shock $\varepsilon_{x,t+1}$. As it is common in the affine term structure models, I add the adjustment for Jensen's inequality $-\frac{1}{2}a_t^2\sigma_x^2$. The price of risk is an affine function of foreign fundamentals. This fundamentals variable, as well as the risk-free rate, follow AR(1) processes whose innovations have a joint normal distribution

$$\boldsymbol{\varepsilon_t} = \begin{bmatrix} \varepsilon_{r,t} \\ \varepsilon_{x,t} \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \sigma_r^2 & \sigma_{rx} \\ \sigma_{rx} & \sigma_x^2 \end{bmatrix}\right) \tag{23}$$

As shown in Lettau and Wachter (2011), allowing these shocks to be *negatively* correlated is important in replicating the average upward-sloping yield curve of the risk-free bonds. To see this, consider the pricing formula for a generic portfolio of zero-coupon riskless bonds with maturity parameter δ. The price of such a portfolio is

$$\begin{aligned} q_t &= \mathbb{E}_t M_{t,t+1}\Big[1+(1-\delta)q_{t+1}\Big] \\ &= \mathbb{E}_t M_{t,t+1}\,\mathbb{E}_t\Big[1+(1-\delta)q_{t+1}\Big] + (1-\delta)cov\Big(M_{t,t+1}, q_{t+1}\Big) \end{aligned}$$

where the second equality comes from applying the definition of covariance. So if $M_{t,t+1}$

is negatively correlated with q_{t+1} then prices of bond portfolios with a longer maturity profile (i.e. ones that have smaller δ) are lower than shorter-term ones. As a result, the risk-free yield curve slopes upwards on average. Why would the bond price be negatively correlated with the lenders' stochastic discount factor? Consider again formulas (19)-(21). Assume that the covariance between innovations to risk-free rate and the US fundamentals σ_{rx} is negative (as I confirm it in the quantitative analysis in Section 6.2). Then, we have that positive shocks to the risk-free rate are associated with higher SDF of the lenders (through their negative covariance with $\varepsilon_{x,t}$), but also lead to lower bond prices (because of steeper discounting of the future).

6.2 Calibrating the lenders' stochastic discount factor

I calibrate the stochastic discount factor using the simulated method of moments. To do so, I first solve for the prices of risk-free zero-coupon bonds at each maturity $n \geq 1$ in closed form (Appendix C shows these derivations). Subsequently, I simulate the behavior of stochastic variables r and x over time. I calculate the corresponding bond prices and obtain the entire yield curve at each simulated time period. From these simulations, I select two sets of targeted moments. First, I estimate the following VAR using the initial 63 periods of each simulated path, corresponding to the time period 1954-2016 for which I have the data on the US yield curves

$$Z_t = A_0 + A\, Z_{t-1} + \Sigma \varepsilon_t \tag{24}$$

where $Z_t = \begin{bmatrix} y_{1,t} \\ y_{10,t} - y_{1,t} \end{bmatrix}$, is a vector containing the yield on 1-year bond and the 10- to 1-year term premium. Σ is the variance-covariance matrix of this regression, A is restricted to be a diagonal matrix containing the persistence parameters of the 1-year yield and the term premium, while A_0 is a [2x1] vector of unconditional means. The values of r_t and x_t in the initial period are set to approximate the 1- and 10-year yields on US bonds in year 1954. I compute average estimates for the parameters of this regression across sufficiently many simulated paths and compare them to the identical regression estimates in the data. These estimates provide me with six targeted moments: persistence and standard deviation of both 1-year yield and the term premium, unconditional mean of the term premium and the correlation between the two. Similarly as in Section 4.3.2, I explicitly *do not* use the unconditional mean of the 1-year yield, as explained in the following paragraph.

The second source of my empirical targets is based on fitting an AR(1) process for the 10-year real yields using a combination of historical data and OECD's long-horizon forecasts as described in Section 4.3.2. The unconditional mean and persistence are inferred by minimizing the sum of squared deviations between the model-generated projection for years 2012-2060 and the empirical one. Then, given these estimates the root mean square error is backed out from historical data for years 1960-2012. To match these estimates, I run autoregressions for the 10-year yield in simulated samples consisting of 101 periods (which corresponds to years 1960-2060). Table 9 summarizes the values of parameters that drive the stochastic discount factor, along with the outcome of moment-matching.

Table 9: Calibration of the lenders' stochastic discount factor

Symbol	**Meaning**	**Value**	**Source**
$\bar{r}$	mean 1-yr rate	0.035	Joint calibration based on 10-year yield AR(1) and 1-year yield + term premium VAR
ρ_r	persist. 1-yr rate	0.92	
σ_r	st.dev. 1-yr rate	0.015	
α_0	price of risk level factor	-3.11	
α_1	price of risk slope factor	16.59	
μ_x	mean US fundament.	0.21	
ρ_x	persist. US fundament.	0.28	
σ_x	st.dev. US fundament.	0.07	
σ_{rx}	corr. 1-yr rate & US fund.	-0.28	
Calibration targets		**Model**	**Data**
Short-sample VAR targets:			
mean US 10/1 yr term premium		0.011	0.01
persist. US 10/1 yr term premium		0.44	0.414
st.dev. US 10/1 yr term premium		0.012	0.012
persist. US 1-yr yield		0.79	0.84
st.dev. US 1-yr yield		0.017	0.018
corr. 1-yr yield & 10/1 yr term premium		-0.574	-0.528
Long-sample regression targets:			
mean US 10-yr yield		0.042	0.042
persistence US 10-yr yield		0.86	0.86
st.dev. US 10-yr yield		0.011	0.01

To summarize, I select the values of nine parameters that govern the behavior of the lenders' pricing kernel by targeting nine empirical moments in an SMM procedure. Six of those come from the estimates of VAR of real short yields and the term premium for 1954-2016, while the remaining three are based on fitting an AR(1) of the 10-year real yield for years 1960-2060. Table 9 summarizes the achieved parameter values and matching of the targets. Notice that the unconditional mean of the one-year real interest rate amounts to 3.5%, significantly above the value that we would back out by using historical data only. Notice also that the average term premium amounts to 1.1% with moderate persistence of 0.44, implying a yield curve that slopes upwards on average with occasional flattening or inversions as in the data. The negative comovement between the one-year yield and the term premium suggests that such inversions tend to occur when interest rate rises and it is obtained through a negative correlation parameter between the underlying variables r and x, as explained above. Figure 12 shows that the model is able to reproduce the empirical series of one-year yield and the term premium with high accuracy.

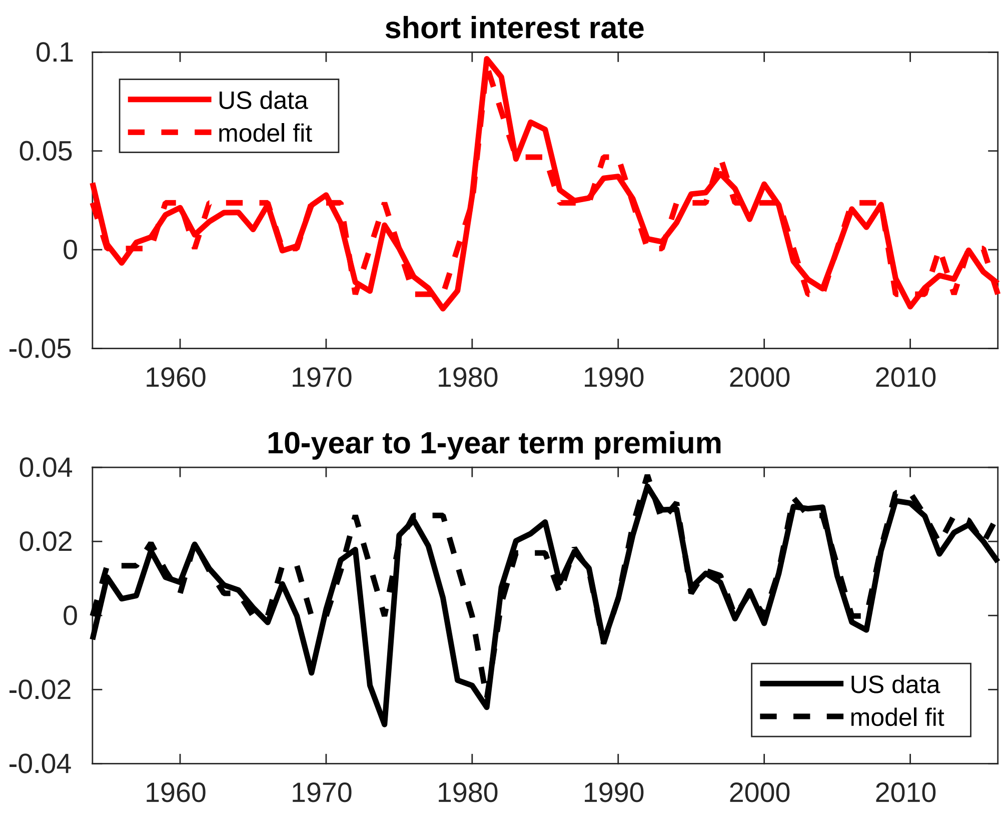


Figure 12: Data vs. model fit of the short yield and the risk premium

6.3 Maturity choices with risk-averse lenders

Having calibrated the lenders' stochastic discount factor, I proceed to analyze how the behavior of the model changes in terms of optimal maturity choices and spread curves. In order to match the targets for Mexico's government behavior, I recalibrate the structural parameters and obtain $\beta = 0.793$ and $\phi = 0.056$. Table 10 summarizes the key model predictions. The top panel shows that the government still finds it optimal to issue a non-negligible share of the ultralong debt, although smaller by 0.5 percentage point relative to the model with risk-neutral lenders. The bottom panel confirms that the core mechanism becomes weaker in this setup through a slightly lower correlations between the risk-free rate and the ultralong shares or the relative hedging term. This is not surprising given that risk-averse lenders generally require a higher compensation for holding longer-dated bonds when the risk-free yield curve slopes upwards on average. The last column also provides correlations with the foreign fundamentals variable. It noteworthy that the ultralong shares tend to increase when the fundamentals weaken and the yield curve flattens out. Overall, it should be concluded that introducing risk-averse lenders does not fundamentally affect the mechanism proposed in this paper.

Table 10: Ergodic moments in the model with risk-averse lenders

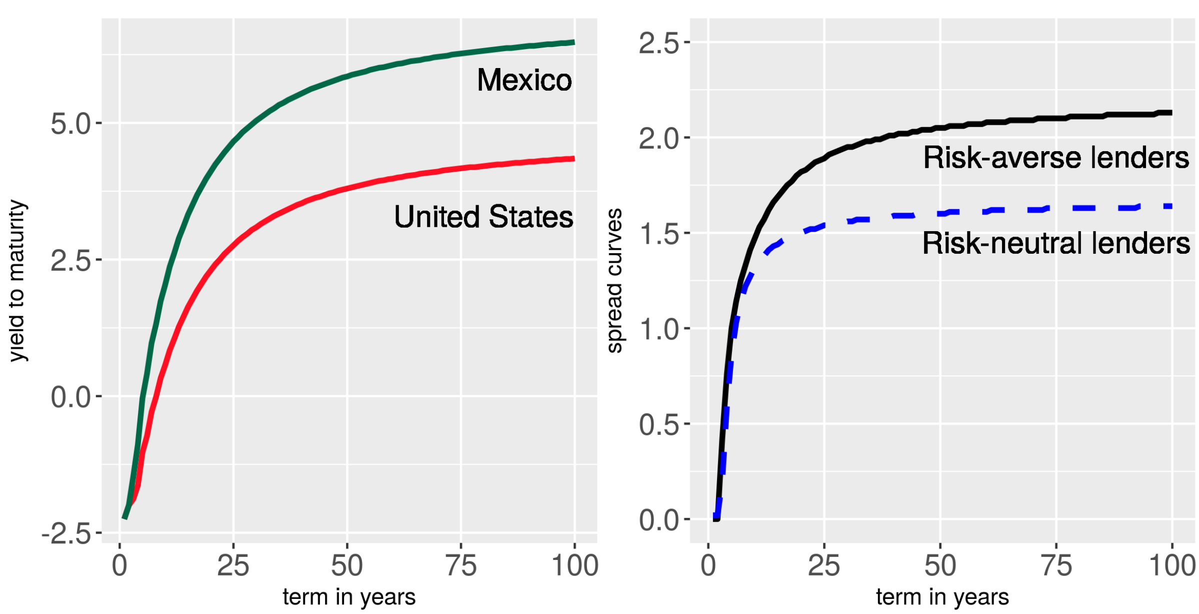
	Mean	**St. dev.**	
ultra share	4.46	1.54	
ultra new share	0.00	3.75	
duration	4.68	0.53	
spread reg.	1.78	0.14	
spread ultra	1.70	0.13	
spread crve reg/ultra	−0.10	0.20	
Correlation with:	r	g	x
ultra share	−0.82	−0.08	−0.33
ultra new share	−0.24	−0.21	−0.45
duration	−0.89	−0.01	−0.23
spread reg	−0.29	−0.61	−0.33
spread ultra	0.91	−0.16	0.13
spread crve reg/ultra	0.76	0.32	0.31
debt/y	−0.96	−0.25	0.21
tb/y	0.49	−0.79	0.14
default prob.	0.01	−0.55	−0.64
default frequency	0.02	−0.36	0.04
rel. incentive	0.08	−0.02	−0.89
rel. hedging	0.28	−0.46	−0.71
u'(c)	0.13	−0.98	0.03

Note: except for duration, the moments in the top panel are given in percentage points.

7 Event analysis: 2010-2015

In this section I deviate from the ergodic distribution of the model and focus on the specific state of the world that led Mexico to issue 100-year bonds. To this end, I compute average values for the outstanding debt, growth rate, risk-free rate and the foreign fundamentals (the series for which is recovered by fitting the paths of the 1-year rate and the 10- to 1-year term premium in Figure 12 in years 2010-2015. I feed these values as states into the model and generate predictions for debt issuance and spreads.

I find that the government chooses to issue 7.4% of its total debt in the form of perpetuities, which is higher than the unconditional average share for this model, and also slightly higher than in reality. Figure 13 illustrates the predicted yield and spread curves for zero-coupon bonds in that state. Panel 13(a) plots the yield curves for both Mexico and the United States (i.e. the risk-free curve). Both curves are strictly increasing with the Mexican curve increasing faster. The resulting spread curve, defined as the difference between the two yield curves, is itself increasing as can be noticed on panel 13(b). The predicted spread on the 100-year zero-coupon bond amounts to 2.13% which falls short of the empirical one of 2.3%, the spread inferred by comparing Mexico's Euro-denominated bond with the Austrian one, as depicted in Figure 2. Interestingly, Figure 13 also in-



(a) Yield curves: MX vs US (b) Spread curves

Figure 13: Yield and spread curves in the model

cludes the spread curve predicted for this state by the benchmark model with risk-neutral lenders. In that case, the spread levels out at 1.64% which shows that investor risk premia are an important component of the spreads on long-dated sovereign bonds. The difference between these curves is due to the fact that shocks to foreign fundamentals are negatively correlated with shocks to risk free rate, which in turn impact default incentives as Figure 7 showed. Then, sovereign bonds are likely to default precisely in the states where foreign lenders' marginal utility is high and thus carry an additional premium.

Because the model calibrated to Mexico's history of sovereign defaults over the last century *underpredicts* the actual spread on 100-year bonds, I interpret it as evidence that financial markets expect Mexico to remain a risky borrower in the foreseeable future.

8 Discussion

In this section, I briefly discuss three additional elements that should be taken into account in the analysis of 100-year bonds, and are potential avenues for future research.

8.1 Demand for ultralong bonds

While data on ultralong bond holders is not easily accessible, it is anecdotally known that the majority of demand for sovereign century bonds came from insurance companies and pension funds. Such firms have liabilities extending decades into the future and often seek assets that provide matching cash flows. Indeed, Greenwood and Vissing-Jorgensen (2018) show that pension and insurance companies' investment in long-dated government bonds tends to depress the 30-10 year yield spread. These findings suggest that the assumption about competitive lenders made in this paper may not be satisfied for such securities. To better understand the demand in this market, one may consider need to a preferred-habitat type of environment in the spirit of Vayanos and Vila (2009).

8.2 Bond maturity and recovery rates

Throughout this paper, I assumed that the recovery rate on both types of debt following a default settlement are equal. This assumption is made due to a lack of evidence on the settlements of bonds with such long maturity. Recently however, Asonuma, Niepelt and Ranciere (2017) show that this may not hold in reality, as longer-term debt tends to have higher recovery rates than shorter term debt. Investigating the effects of differential recovery rates on optimal maturity choice is left as avenue for future research.

8.3 Liquidity of ultralong bonds

A final remark concerns the effect of liquidity frictions. Traditional models of sovereign debt ignore the trading process assuming that investors who hold bonds are always able to sell them at competitive prices. In reality, such investors need to find a counter-party willing to purchase the bond from, which involves transaction costs and gives rise to a bid-ask spread. For most commonly traded bonds, such spreads are low and can plausibly be ignored. This is not necessarily the case for 100-year bonds which attract a certain type of investors (see Section 8.1). Such a market may not be equally liquid and the resulting liquidity spread will add to the observed yields on top of the default and term premia, which we have modeled so far.

Table 11 presents the average bid-ask spread for Mexico's USD-denominated century bond, along with the one for a more typical bond with maturity below 10 years. The last column contains the relative spread, calculated as $\frac{p^A - p^B}{0.5(p^A + p^B)}$ and expressed in basis points. The spread on the 100-year bond is 85 basis points on average, 2.5 times the spread on regular maturity bond. This suggests that liquidity premia are potentially an important component of the total observed spread.

Table 11: Bid-ask spreads for Mexican bonds: 2013-2018

Bond	Average yield	Average bid-ask spread
Bond expiring in 2110	568.4	85.2
Bond expiring in 2023	349	33

Note: Yields and spreads averaged over the time period 2013-2018 and expressed in basis points.

9 Conclusion

This paper builds a quantitative model of sovereign debt to understand the recent issuances of century bonds among the countries with a history of serial default. The main hypothesis, supported by forecast evidence from long-term projections, is that countries issue ultralong debt to hedge against low-frequency movements in the risk-free interest rate. The calibrated model predicts that issuances predominantly occur when the risk-free rate is low and expected to rise. This is due to the fact that the elevated default risk stemming from a possible rate increase makes such states more risky and their future outcomes more volatile; hence raising the demand for insurance.

The model augmented with risk-averse lenders generates a realistic share of the ultralong bond issuance in years 2010-2015. The predicted spread on 100-year bonds falls slightly short relative to what can be inferred from the available evidence. Hence, through the lens of the model I find little support for the common view that Mexico obtained better credit conditions on 100-year bonds than its prior default history would dictate.

References

AGUIAR, MARK AND GITA GOPINATH (2006): "Defaultable debt, interest rates and the current account", *Journal of International Economics* 69, 64-83.

ALMEIDA, VICTOR, CARLOS ESQUIVEL, TIMOTHY KEHOE, AND JUAN PABLO NICOLINI (2019): "Did the 1980s in Latin America Need to Be a Lost Decade?", *unpublished manuscript*.

AGUIAR, MARK, MANUEL AMADOR, HUGO HOPENHAYN, AND IVAN WERNING (2019): "Take the Short Route: Equilibrium Default and Debt Maturity", *Econometrica*, 87, 423-462.

ARELLANO, CRISTINA (2008): "Default Risk and Income Fluctuations in Emerging Economies", *American Economic Review*, 98, 690-712.

ARELLANO, CRISTINA AND ANANTH RAMANARAYANAN (2012): "Default and Maturity Structure in Sovereign Bonds", *Journal of Political Economy*, 120, 187-232.

ASONUMA, TAMON, DIRK NIEPELT, AND ROMAIN RANCIERE (2017): "Sovereign Bond Prices, Haircuts and Maturity", *NBER Working Paper*, 23864.

BORRI, NICOLA AND ADRIEN VERDELHAN (2011): "Sovereign Risk Premia", *unpublished manuscript*.

CHATTERJEE, SATYAJIT AND BURCU EYIGUNGOR (2012): "Maturity, Indebtedness, and Default Risk", *American Economic Review*, 102, 2674-2699.

CHEN, SOPHIA, PAOLA GANUM, LUCY LIU, LEONARDO MARTINEZ, AND SOLEDAD MARTINEZ PERIA (2018): ”Debt Maturity and the Use of Short-term Debt: Evidence from Sovereigns and Firms”, *IMF Research Department Paper Series*, 19/03.

EATON, JONATHAN AND MARK GERSOVITZ (1981): “Debt with potential repudiation”, *Review of Economic Studies* 48, 289-309.

GREENWOOD, ROBIN, AND ANNETTE VISSING-JORGENSEN (2018): “The Impact of Pensions and Insurance on Global Yield Curves”, *unpublished manuscript*.

GUIMARAES, BERNARDO (2011): “Sovereign default: Which shocks matter?”, *Review of Economic Dynamics* 14, 553-576.

HATCHONDO, JUAN CARLOS AND LEONARDO MARTINEZ (2009): “Long-duration bonds and sovereign defaults”, *Journal of International Economics*, 79, 1717-125.

HATCHONDO, JUAN CARLOS, LEONARDO MARTINEZ AND HORACIO SAPRIZA (2010): “Quantitative properties of sovereign default models: Solution methods matter”, *Review of Economic Dynamics*, 13, 919-933.

HATCHONDO, JUAN CARLOS, LEONARDO MARTINEZ AND CESAR SOSA-PADILLA (2016): “Debt Dilution and Sovereign Default Risk”, *Journal of Political Economy*, 124, 1383-1422.

INEGI, (1995): “Estadísticas Históricas de México”, México D.F.

JOHRI, ALOK, SHAHED K. KHAN, AND CESAR SOSA-PADILLA (2022): “Interest Rate Uncertainty and Sovereign Default Risk”, *Journal of International Economics*, 139, 103681.

LETTAU, MARTIN AND JESSICA A. WACHTER (2011): “The term structures of equity and interest rates”, *Journal of Financial Economics*, 101, 90-113.

LONGSTAFF, FRANCIS A., JUN PAN, LASSE H. PEDERSEN, AND KENNETH J. SINGLETON (2011): “How Sovereign is Sovereign Credit Risk?”, *American Economic Journal: Macroeconomics*, 3(2), 75-103.

PALUSZYNSKI, RADOSLAW (2023): “Learning About Debt Crises”, *American Economic Journal: Macroeconomics*, 15(1): 1-30.

REINHART, CARMEN M. AND KENNETH S. ROGOFF (2009): “This Time is Different: Eight Centuries of Financial Folly”, *Princeton University Press*.

STANDARD & POOR'S (2014): "Sovereign Rating and Country T&C Assessment Histories", *McGraw Hill Financial*, February 5, 2014.

TOURRE, FABRICE (2017): "A Macro-Finance Approach to Sovereign Debt Spreads and Returns", *unpublished manuscript*

VAYANOS, DIMITRI, AND JEAN-LUC VILA (2009): "A Preferred-Habitat Model of the Term Structure of Interest Rates", *NBER Working Paper*, 15487.

Appendix

A Detrended model

The model presented in Section 3 is non-stationary and thus cannot be computed directly. I detrend the model by normalizing all allocation variables by the last period income level, i.e. $z \equiv \frac{Z}{Y_{-1}}$, where $Z \in \{C, B, B_u\}$. The normalized vector of stochastic state variables then boils down to $s \equiv \{g, r\}$, and the detrended budget constraint takes the form

$$c_t = g_t - b_t - b_{u,t}\kappa + q_t\big(b_{t+1}g_t - (1-\delta)b_t\big) + q_{u,t}\big(b_{u,t+1}g_t - b_{u,t}\big) \tag{25}$$

To normalize the government's value functions, I guess and verify that $v^j(B, B_u, \mathbf{S}) = Y_{-1}^{1-\gamma} v^j(b, b_u, \mathbf{s})$, $j \in \{r, d\}$. The detrended value associated with repayment becomes

$$v^r(b, b_u, \mathbf{s}) = \max_{c, b', b'_u} \Big\{ u(c) + \beta\, g^{1-\gamma} \mathbb{E}\Big[\, v(b', b'_u, \mathbf{s}') | \mathbf{s} \Big] \Big\} \tag{26}$$

subject to the budget constraint (25), while the detrended value associated with default is

$$\begin{aligned} v^d(b, b_u, \mathbf{s}) = {} & u\Big(g(1-\phi)\Big) \\ & + \beta\, g^{1-\gamma} \mathbb{E}\Big[\theta\, v\big(\omega(1+r)b, \omega(1+r)b_u, \mathbf{s}'\big) + (1-\theta)\, v^d\big((1+r)b, (1+r)b_u, \mathbf{s}'\big) | \mathbf{s}\Big] \end{aligned} \tag{27}$$

The government's choice whether to default or not is normalized to

$$v(b, b_u, \mathbf{s}) = \max_{d \in \{0,1\}} \Big\{ (1-d) v^r(b, b_u, \mathbf{s}) + d v^d(b, b_u, \mathbf{s}) \Big\} \tag{28}$$

The prices of current and defaulted bonds with regular maturity are

$$q(b', b'_u, \mathbf{s}) = \frac{1}{1+r}\, \mathbb{E}\Big[d\left(b', b'_u, \mathbf{s}'\right)\, q^d\left(b', b'_u, \mathbf{s}'\right) + \Big(1 - d(b', b'_u, \mathbf{s}')\Big)\Big[1 + (1-\delta) q(b'', b''_u, \mathbf{s}')\Big] \mid \mathbf{s}\Big] \tag{29}$$

$$\begin{aligned} q^d(b, b_u, \mathbf{s}) = \mathbb{E}\Big[&(1-\theta)\, q^d\Big((1+r)b, (1+r)b_u, \mathbf{s}'\Big) \\ &+ \theta\omega\left[1 - d\big(\omega(1+r)b, \omega(1+r)b_u, \mathbf{S}'\big)\right]\Big[1 + (1-\delta) q(b', b'_u, \mathbf{s}')\Big] \\ &+ \theta\omega\, d\Big(\omega(1+r)b, \omega(1+r)b_u, \mathbf{s}'\Big)\, q^d\Big(\omega(1+r)b, \omega(1+r)b_u, \mathbf{s}'\Big) \mid \mathbf{s}\Big] \end{aligned} \tag{30}$$

and analogously for the ultralong bonds.

B Deriving the bonds portfolio

In this section I show that a portfolio of zero-coupon bonds with face values $\{b_{n,t}\}$ where n is maturity such that $b_{n,t} = (1-\delta) b_{n-1,t}$ for all $n \geq 1$ can be represented by a single state variable B_t which satisfies $b_{n,t} = (1-\delta)^{n-1} B_t$ for all $n \geq 1$. For illustration, I will ignore the ultralong debt in this derivation and assume that the recovery rate of debt is zero.

Let $q_n(b', \mathbf{S})$ be the price of a zero-coupon bond with maturity n that depends on current realizations of the stochastic state variables $\mathbf{S}$. Prices of such bonds follow the recursion

$$q_n(b', \mathbf{S}) = \mathbb{E}_{\mathbf{S}'|\mathbf{S}}\Big[M(\mathbf{S}, \mathbf{S}')\big(1 - d(B', \mathbf{S})\big) q_{n-1}(B'', \mathbf{S}')\Big] \tag{31}$$

Let $Q(B', \mathbf{S})$ be the price of a long-term bond as in Chatterjee and Eyigungor (2012) that pays a declining sequence of coupons over time: $1, 1-\delta, (1-\delta)^2, \ldots$. This price satisfies

$$Q(B', \mathbf{S}) = \mathbb{E}_{\mathbf{S}'|\mathbf{S}} M(\mathbf{S}, \mathbf{S}')\big(1 - d(B', \mathbf{S})\big)\Big[1 + (1-\delta) Q(B'', \mathbf{S}')\Big] \tag{32}$$

Now consider a portfolio of zero-coupon bonds whose face value declines with maturity at rate δ. The price of such a portfolio is

$$\begin{aligned}
&\sum_{n=1}^{\infty}(1-\delta)^{n-1}q_n(B',\mathbf{S}) \\
&= q_1(B',\mathbf{S}) + \sum_{n=2}^{\infty}(1-\delta)^{n-1}\mathbb{E}_{\mathbf{S}'|\mathbf{S}}\Big[M(\mathbf{S},\mathbf{S}')\big(1-d(B',\mathbf{S})\big)q_{n-1}(B'',\mathbf{S}')\Big] \\
&= q_1(B',\mathbf{S}) + \mathbb{E}_{\mathbf{S}'|\mathbf{S}}\Big[M(\mathbf{S},\mathbf{S}')\big(1-d(B',\mathbf{S})\big)\sum_{n=2}^{\infty}(1-\delta)^{n-1}q_{n-1}(B'',\mathbf{S}')\Big]
\end{aligned}$$

Denote $k \equiv n-1$:

$$\begin{aligned}
&= q_1(B',\mathbf{S}) + \mathbb{E}_{\mathbf{S}'|\mathbf{S}}\Big[M(\mathbf{S},\mathbf{S}')\big(1-d(B',\mathbf{S})\big)\sum_{k=1}^{\infty}(1-\delta)^{k}q_{k}(B'',\mathbf{S}')\Big] \\
&= q_1(B',\mathbf{S}) + (1-\delta)\mathbb{E}_{\mathbf{S}'|\mathbf{S}}\Big[M(\mathbf{S},\mathbf{S}')\big(1-d(B',\mathbf{S})\big)\sum_{k=1}^{\infty}(1-\delta)^{k-1}q_{k}(B'',\mathbf{S}')\Big] \\
&= q_1(B',\mathbf{S}) + (1-\delta)\mathbb{E}_{\mathbf{S}'|\mathbf{S}}\Big[M(\mathbf{S},\mathbf{S}')\big(1-d(B',\mathbf{S})\big)Q(B'',\mathbf{S}')\Big] \\
&= \mathbb{E}_{\mathbf{S}'|\mathbf{S}}M(\mathbf{S},\mathbf{S}')\big(1-d(B',\mathbf{S})\big)\Big[1+(1-\delta)Q(B'',\mathbf{S}')\Big] \qquad (33)
\end{aligned}$$

which is the recursive definition of the bond price in formula (32).

C Pricing the zero-coupon risk-free bonds

In this section I provide the closed-form solutions for zero-coupon risk-free bond prices which are used to calibrate the lenders' stochastic discount factor. Let $q^{rf}_{n,t}$ denote the logarithm of a risk-free zero-coupon bond that matures in n periods, and let $M_{t,t+1}$ be the pricing kernel. The prices of such bonds satisfy

$$exp\{q^{rf}_{n,t}\} = \mathbb{E}_t\Big(M_{t,t+1}exp\{q^{rf}_{n-1,t+1}\}\Big)$$

Let the [2x1] vector of factors Z_t follow a joint autoregressive process

$$Z_{t+1} = \mu + \Phi Z_t + \Sigma\varepsilon_{t+1}$$

Following the notation from Campbell (2018), we will define the one-period yield as

$$y_{1,t} = \delta_0 + \delta_1' Z_t$$

and the pricing kernel takes the general form of

$$M_{t,t+1} = exp\Big\{-y_{1,t} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon_{t+1}\Big\}$$

where $\lambda_t = \lambda_0 + \lambda_1 Z_t$. As shown in Campbell (2018), the zero-coupon bond prices have a closed-form solution of the form

$$q_{n,t} = A_n + B_n' Z_t$$

where

$$\begin{aligned} A_{n+1} &= -\delta_0 + A_n + B_n'(\mu - \Sigma\lambda_0) \\ B_{n+1} &= (\Phi - \Sigma\lambda_1)' B_n - \delta_0 \\ A_1 &= -\delta_0 \\ B_1 &= -\delta_1 \end{aligned}$$

The zero-coupon yields can be then obtained as

$$y_{n,t} = -\frac{A_n}{n} - \frac{B_n'}{n} Z_t$$

In my application, the two-factor vector of states $Z_t = [r_t \ x_t]'$ consists of the risk-free rate and the foreign fundamentals variable. The law of motion equations (20)-(21) for these two factors impose the following conditions on the auxiliary terms

$$\delta_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \lambda_0 = \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix}$$

$$\delta_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda_1 = \begin{bmatrix} 0 & 0 \\ 0 & \alpha_1 \end{bmatrix}$$