

# Ladder Operators

In QM and in Matrix mechanics, it is commonplace to search for Eigenstates of an operator and the associated eigen values. Herein, matrix methods will be used to demonstrate the ideas, which in turn will be applied to the Harmonic Oscillator of QM.

Given a Matrix  $[M]$  which operates on a vector  $|X\rangle$  to create a new vector  $|Y\rangle$ . The mathematical expression is given by the algebraic equation:

$$[M] \circ |X\rangle = |Y\rangle.$$

In certain circumstances, the new vector  $|Y\rangle$  is proportional to  $|X\rangle$  with the proportionality constant  $\lambda_n$ . For these special cases,

$$[M] \circ |X_n\rangle = \lambda_n |X_n\rangle.$$

The vectors  $|X_n\rangle$  that satisfy this equation are called eigen vectors (state vectors in QM) and the proportionality constants are called eigen values  $\lambda_n$ . The "line of action" of the eigen vector is not changed when the operator  $[M]$  is applied to it. The line of action is an invariant of the Operator M. (However, the "length" of the vector changes.)

How many eigen vectors and eigen values are there? For  $n \times n$  matrices, there can be as many as  $n$  distinct eigen vectors with up to  $n$  eigen values. Not all of the eigen values are distinct, even though the eigen vectors are distinct. When the eigenvalues admit more than 1 eigen vector, then the states are said to be degenerate.

The eigen vectors of different matrix operators need not be the same, but they can be identical to within a factor.

Sometimes it is possible to find another operator (matrix)  $[a]$  and its "adjoint"  $[a^\dagger]$  such that

$$[M] \circ [a] \circ |X_n\rangle = \lambda_{n+1} |X_{n+1}\rangle \quad \text{and} \quad [M] \circ [a^\dagger] \circ |X_n\rangle = \lambda_{n-1} |X_{n-1}\rangle$$

for the eigen vectors  $|X_n\rangle$  of the Matrix operator  $[M]$ . If this operator can be found, then given one eigen vector, all the other eigen vectors can be deduced.

For example, the harmonic oscillator states are given by the formulas

$$\Psi_n \approx \left( \frac{1}{2^{n/2}} \right) H_n(y) \cdot e^{-y^2/2}$$

where the functions  $H_n(y)$  are the Hermite polynomials.

$$H_0(y) = 1$$

$$H_1(y) = 2y$$

$$H_2(y) = 4y^2 - 2$$

$$H_3(y) = 8y^3 - 12y$$

$$H_4(y) = 16y^4 - 48y^2 + 12$$

etc.

So the ground state is of the format:  $\Psi_0 = C_0 e^{-y^2/2}$

Note that the operate  $a = (y - d/dy)/\sqrt{2}$  acting on  $\Psi_0$  yields

$$[a] \cdot \Psi_0 = \{(y - d/dy)/\sqrt{2}\} \Psi_0 e^{-y^2/2} = C_1 \cdot 2y \cdot e^{-y^2/2} = \Psi_1$$

$$[a] \cdot \Psi_1 = [a] \circ [a] \circ \Psi_0 = \{(y + d/dy)/\sqrt{2}\} \Psi_1 = C_2(2y^2 + 2y^2 - 2)e^{-y^2/2} = \Psi_2$$

etc.

Hence  $(y - d/dy)/\sqrt{2}$  is a raising operator.  $(y + d/dy)/\sqrt{2}$  is a lowering operator.