ECE 6364 Spring 2016 HW 07 due 3/8

Problem 1. Show that the continuous LaPlacian operator $\nabla^2 f(x,y)$ is rotationally invariant; ; i.e. given a set of axes x', y' rotated by any angle θ from axes x, y, it holds that $\frac{\partial^2 f(x,y)}{\partial^2 x'} + \frac{\partial^2 f(x,y)}{\partial^2 y'} = \frac{\partial^2 f(x,y)}{\partial^2 x} + \frac{\partial^2 f(x,y)}{\partial^2 y}$

Hint: Note that partial derivative $\frac{\partial^n}{\partial x^k \partial y^{n-k}}[\cdot]$ is a linear operator. That is, given any scalars a_1, a_2 and two functions $f_1(x,y)$, $f_2(x,y)$, it holds $\frac{\partial^n}{\partial x^k \partial y^{n-k}}[a_1 f_1(x,y) + a_2 f_2(x,y)] = a_1 \frac{\partial^n}{\partial x^k \partial y^{n-k}}[f_1(x,y)] + a_2 \frac{\partial^n}{\partial x^k \partial y^{n-k}}[f_2(x,y)]$.

We would like to examine which partial derivatives are rotation-invariant. Given coordinate system (x,y), define a rotated coordinate system (x_θ, y_θ) related to (x,y) by $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_\theta \\ y_\theta \end{bmatrix}$ and $\begin{bmatrix} x_\theta \\ y_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

A partial derivative operator is rotation-invariant if $\frac{\partial^n}{\partial x^k \partial y^{n-k}}[f(x,y)] = \frac{\partial^n}{\partial (x_\theta)^k \partial (y_\theta)^{n-k}}[f(x,y)]$.

Note $\frac{\partial}{\partial x_\theta}[f(x,y)] = \frac{\partial}{\partial x}[f(x,y)]\frac{\partial}{\partial x_\theta}[x] + \frac{\partial}{\partial y}[f(x,y)]\frac{\partial}{\partial x_\theta}[y] = \frac{\partial}{\partial x}[f(x,y)]\cos(\theta) + \frac{\partial}{\partial y}[f(x,y)]\sin(\theta)$.

Likewise $\frac{\partial}{\partial y_\theta}[f(x,y)] = \frac{\partial}{\partial x}[f(x,y)]\frac{\partial}{\partial y_\theta}[x] + \frac{\partial}{\partial y}[f(x,y)]\frac{\partial}{\partial y_\theta}[y] = -\frac{\partial}{\partial x}[f(x,y)]\sin(\theta) + \frac{\partial}{\partial y}[f(x,y)]\cos(\theta)$.

For non-zero angles θ, $\frac{\partial}{\partial x_\theta}[f(x,y)] \neq \frac{\partial}{\partial x}[f(x,y)]$, $\frac{\partial}{\partial y_\theta}[f(x,y)] \neq \frac{\partial}{\partial y}[f(x,y)]$, and $\frac{\partial}{\partial x_\theta}[f(x,y)] + \frac{\partial}{\partial y_\theta}[f(x,y)] \neq \frac{\partial}{\partial x}[f(x,y)] + \frac{\partial}{\partial y}[f(x,y)]$, so these are all rotation-variant. However, the LaPlacian operator $\nabla^2[f(x,y)] = \frac{\partial^2}{\partial x^2}[f(x,y)] + \frac{\partial^2}{\partial y^2}[f(x,y)]$ is rotation-invariant.

Problem 2.

Find the convolution-mask $\underline{\mathbf{M}}$ that will implement the discrete derivative $\Delta_x^5 \Delta_y^4 f_{i,j}$ as $\Delta_x^5 \Delta_y^4 f_{i,j} = \underline{\mathbf{M}} ** \{f_{i,j}\}$

Problem 3. You want to both equalize and sharpen an image. Does it matter which you do first? Why or why not?

Problem 4. Digital image (1) below shows a step-edge in the presence of noise.
Digital image (2) below shows a bright vertical line against a noisy dark background.

(A) Apply 1st-order neighborhood averaging to both images.
(B) Apply 1st-order neighborhood median filtering to both images.
(C) Which performs better at reducing the noise while preserving the step edge?
(D) Which performs better at reducing the noise while preserving the line?

(1) a step-edge in the presence of noise.

1	2	12	11	10
2	0	11	10	12
1	1	10	12	10
0	1	11	11	10
2	0	11	10	12

(2) a bright vertical line against a noisy dark background.

1	2	12	2	0
2	0	11	1	1
1	1	10	1	1
0	1	11	1	0
2	0	11	0	2