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The consumption CAPM (CCAPM).

There are many versions of the consumption CAPM, depending on the assumed utility function of consumers. The choice made in Romer is one, but I don't like that choice much since the CCAPM relation that he derives depends on the parameter of the utility function in a way that I find hard to interpret.

The Euler eq. is

$$U'(C_t) \;=\; E_t\{U'(C_{t+1}) * \frac{1+R_{it+1}}{1+\rho}\} \;. \tag{1}$$

We will derive an asset pricing equation resembling the CAPM from starting from the Euler Eq. subject to some assumptions. The first assumption is that $U(C_t) = \ln(C_t)$ which implies that $U'(C_t) = 1/C_t$. From (1) we then get

$$\frac{1}{C_t} = E_t\{\frac{1}{C_{t+1}}\frac{1+R_{it+1}}{1+\rho}\} \;\Leftrightarrow E_t\{\frac{C_t}{C_{t+1}}\frac{1+R_{it+1}}{1+\rho}\} = 1 \tag{2}$$

Equation (2) holds for any asset, so in particular it holds for the safe asset giving

$$E_t\{\frac{C_t}{C_{t+1}}\frac{1+R_F}{1+\rho}\} = 1 \tag{3}$$

Now subtract (3) from (2) and you get $E_t\{\frac{C_t}{C_{t+1}}\frac{R_{it+1}-R_F}{1+\rho}\} = 0$. Since $C_t/C_{t+1} = -(C_{t+1} - C_t)/C_{t+1} + 1$ we get

$$-(E_t R_{it+1} - R_F) = -E_t\{\frac{C_{t+1}-C_t}{C_{t+1}}(R_{it+1} - R_F)\} \;,$$

(after killing the $(1+\rho)$ denominator on each side) which is approximately

$$E_t R_{it+1} - R_F = E_t\{\frac{C_{t+1}-C_t}{C_t}(R_{it+1} - R_F)\} \;. \tag{4}$$

You don't actually need to make this approximation but since $\frac{C_{t+1}-C_t}{C_t}$ is the usual measure of the growth-rate in consumption, I prefer the form (4). Now assume that some asset c^* exists such that the return R_{c^*t+1} is equal to the growth rate in consumption. Is this a reasonable assumption? Not sure, but that gives us a CAPM-like formula. Eq. (4) will then have form

$$E_t R_{it+1} - R_F = E_t\{R_{c^*t+1}(R_{it+1} - R_F)\} \ , \tag{5}$$

or (moving back one period to simplify notation)

$$E_t R_{it} - R_F = E_t\{R_{c^*t}(R_{it} - R_F)\} \ . \tag{6}$$

Here, by the law of iterated expectations, we can take the unconditional expectation and drop the t-subscript going forward. And because in general $EXY = Cov(X,Y) + EXEY$ we get

$$ER_{it} - R_F = Cov(R_{c^*t}, R_{it} - R_F) + E(R_{c^*t})E(R_{it} - R_F) \ , \tag{7}$$

which can be transformed to

$$ER_{it} - R_F = \frac{1}{1 - ER_{c^*t}} Cov(R_{c^*t}, R_{it} - R_F) \ . \tag{8}$$

Note the important interpretation of (8): An asset that has a positive correlation with the growth rate of consumption will have a high return, while an asset that has a negative correlation with consumption will have a low return. Does this make sense? The answer is—yes, it makes perfect sense. An asset that pays off when consumption is low is better to have, since it (partly) provides *insurance* against bad outcomes. And therefore it will have a higher price for a given average payout—and of course a high price implies a low return. So far we have only used the assumption that the growth rate of consumption is an asset in order to simplify notation, but now we use it in earnest. Since (8) has to hold for any asset it also has to hold for c^*. We then get

$$ER_{c^*t} - R_F = \frac{1}{1 - ER_{c^*t}} Var(R_{c^*t}) \ . \tag{9}$$

Divide eq. (8) by (9) and (suppressing the index t in the variance and covariance) you get

$$\frac{ER_{it} - R_F}{ER_{c^*t} - R_F} = \frac{Cov(R_{c^*}, R_i - R_F)}{Var(R_{c^*})} \ . \tag{10}$$

If we denote the coefficient from a regression of R_i on the growth rate of consumption by β_i^c we get

$$ER_{it} - R_F = \beta_i^c * (ER_{c^*t} - R_F) \quad \text{or} \quad R_{it} - R_F = \beta_i^c * (R_{c^*t} - R_F) + u_{it}. \tag{11}$$

This last expression is the CCAPM and β_i^c is called the "consumption beta". (Those names are obviously chosen due to the similarity with the standard CAPM relation.)

Recall that the Euler equation for a general pricing kernel is $E\{r_i - r^f\} = -R^f\, Cov(r_i m_{t+1})$. So apart from normalization, the approximate pricing kernel here the $--$consumption growth/(variance of consumption growth), where the minus comes from the fact that marginal utility varies inversely with the level of consumption. For the "regular" CAPM, the approximate pricing kernel is $--$market return/(variance of market return). Finance people may think of the return as the end goal, but economists would think of the consumption generated by the return. In either event, asset prices are determined by the supply and demand of traders, which probably is why the CAPM fits the data better although it is less directly based on the Euler equaion.

For a test of the CCAPM versus the CAPM see N.G. Mankiw and M.. Shapiro (1986): "Risk and Return: Consumption Beta Versus Market Beta," *The Review of Economics and Statistics* pp. 452-459.

Pricing a payoff with the CCAPM

If we consider the payoff to an asset that you can buy today and which pays off tomorrow, then if you know the consumption beta of that asset, you can calculate the price. This is why the CCAPM is also a capital asset *pricing* model as is the standard CAPM. Here I assume a 2 period model, since this avoids the problem that the price today also will depend on the price tomorrow, which depends on the future payoff and the future price etc. You will try that in the homework.

Consider an asset i with a payoff tomorrow of PO_i. Assume that (average) consumption tomorrow does not depend on how much you invest in the asset—note, that this makes sense when you consider pricing assets using aggregate consumption, but that you have to take into account that your consumption tomorrow is a function of you investments if you use the CCAPM to determine how much to invest given your own personal consumption. The return to a dollar investment in asset i is now

$$R_i = \frac{PO_i}{P_i} - 1 \qquad (a) \ .$$

Assume that you know that the correlation of PO_i with tomorrow's consumption growth R_C is known and equal to $Cov(PO_i, R_C)$. How does this relate to the consumption beta, β_i^c? We have

$$\beta_i^c = \frac{Cov(R_i, R_C)}{Var(R_C)} = \frac{Cov(PO_i/P_i, R_C)}{Var(R_C)} = \frac{1}{P_i}\frac{Cov(PO_i, R_C)}{Var(R_C)} \qquad (b) \ .$$

The CCAPM predicts that $E(R_i - R_F) = \beta_i^c * E(R_C - R_F)$, where R_C is the growth rate in consumption, and if we substitute (a) and (b) into this relation, we find

$$\frac{E\{PO_i\}}{P_i} - 1 - R_F = \beta_i^c * E(R_C - R_F) = \frac{1}{P_i}\frac{Cov(PO_i, R_C)}{Var(R_C)} * E(R_C - R_F) \ .$$

Multiply by P_i on both sides and you get

$$E\{PO_i\} - P_i * (1 + R_F) = \frac{Cov(PO_i, R_C)}{Var(R_C)} E(R_C - R_F) \ ,$$

which you can solve for P_i and get

$$P_i = \frac{E\{PO_i\}}{1 + R_F} - \frac{Cov(PO_i, R_C)}{Var(R_C)} * \frac{E(R_C) - R_F}{1 + R_F} \ .$$

Let us check if this equation makes sense. If the expected payoff is equal to 1 plus the safe return and the asset is not correlated with consumption, then the price is 1. This makes sense. If the asset has payoff that is negatively correlated with the growth rate of consumption then the asset price should be higher, since it provides some element of insurance. [NOTE: some times we say "consumption" rather than "growth rate of consumption" for brevity.] This also follows from the equation, so it makes perfect sense. Example: If the safe interest rate is 2% and the asset has an expected payoff of 20 $, and it is not correlated with consumption, then its price should be

$20/1.02 = 19.61$ \$. Example: Still assume a safe interest rate of 2%. Now assume that the payoff has an expected value of 20 but a covariance with consumption of 18. Assume that the variance of consumption is 3, and that the expected growth rate of consumption is 5%. The price of this asset should be

$$P_i = \frac{20}{1.02} - \frac{18}{3}\frac{0.05 - .02}{1.02} = 19.61 - 6 * 0.0294 = 19.43 \ .$$

The expected return is 20/19.43 -1 = 1.029 %. The fact that the asset has a payoff that is positively correlated with consumption makes the price lower and the asset therefore has a return that is higher than the safe return.