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An introduction to the CAPM model.

We will first sketch the efficient frontier and how to derive the Capital Market Line and we will then derive the CAPM model. An easy-to-read recent article about the CAPM is "An Asset Allocation Puzzle" by Niko Canner, Gregory Mankiw, and David Weil (CMW), in the AER March 1997, pp. 181–191 [available in JSTOR]).

You can also read about the CAPM in any undergraduate (or graduate) finance text. For example, Bodie, Kane, and Marcus "Investments."

Before we go further we will discuss the assumptions under which the CAPM is derived. The following 10 assumptions are sometimes listed. (Other authors may set the assumptions up slightly differently and may state slightly more or less than 10 assumptions; see for example CMW. The content of the set of assumptions is, however, the same). I list the 10 assumptions very briefly, but also include my own comments in []s.

1. No transactions cost. [May be a reasonable approximation for the "large" agents (pension funds etc) who are most important for asset price formation. If we include real estate and other non-financial assets in the model, this assumption may become critical. In general, most asset pricing models ignore illiquid assets, but it is hard to know if that is reasonable. For example, do consumers which high value of their real estate take more financial risk?]

2. Assets are infinitely divisible. [Probably innocuous for financial assets if not literally true.]

3. No income taxes. [Certainly will make pricing of tax-exempt securities crazy (relative to other assets) but that could probably be fixed easily but concentrating on after-tax returns for the typical inverstor. Not obvious how critical it is for other securities.]

4. Single agents can not affect prices. [Some big pension funds may be able to, but under normal circumstances they probably do not, except for short term effects when they unload a big holding of an equity].

5. Investors care only about mean and variance of their total financial portfolio *or* (equivalently) asset returns follow the normal distribution. [A critical assumption and it is obviously wrong that investors care only about mean and variance. Consider the income from an insurance company (left skewed) and a lottery (right skewed) – most people would prefer a lottery even if the mean and variance were equal. Note, however, it is only the mean and variance of the *overall* portfolio that matters and; also note that "caring about mean and variance" is the same as caring about mean and standard deviation and the CAPM usually compare mean and standard deviations.

6. Short sales allowed. [Important?]

7. Unlimited lending and borrowing at the riskless rate. [Not true for me and even big players has to pay a spread between borrowing and lending.]

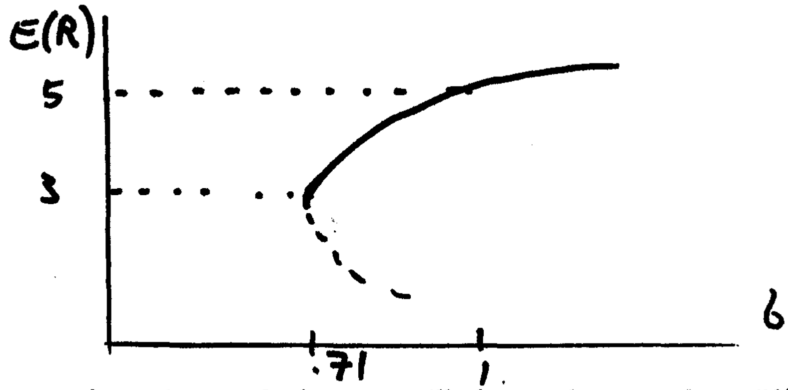
8. All investors have identical expectations. In other word they agree on future values of the mean

and variance of returns. [Not likely to describe the world, but the CAPM may work fine with the average of peoples expectations.]
9. All investors have the same time horizon (and it is a one-period model). [Not true, and may cause the CAPM to work better for (say) monthly returns than for annual returns, or vice versa.]
10. All assets are marketable. [The main exception is human capital (i.e. the sum of future earnings). If agents use asset-markets to hedge against income risk then returns may vary with the business cycle in a way that is not described by the CAPM.]

I list all these assumptions since it is the breakdown of one of more of these that will create deviations from the CAPM. (And for many of them also from the consumption-CAPM and even the Euler equation.) The brief comments that I supplied to the assumptions are only suggestive and you can use your own common sense. Literally the assumptions are of course false, but which of these are critical and how this affect the validity of the CAPM is still very much an open research area.

The central concept underlying the CAPM is the *efficient frontier* developed in 1952 by Harry Markowitz. Markowitz was awarded the 1990 Nobel price for his efforts, together with William Sharpe—for being one of the developers of the CAPM model, and Merton Miller—for contributions to corporate finance.

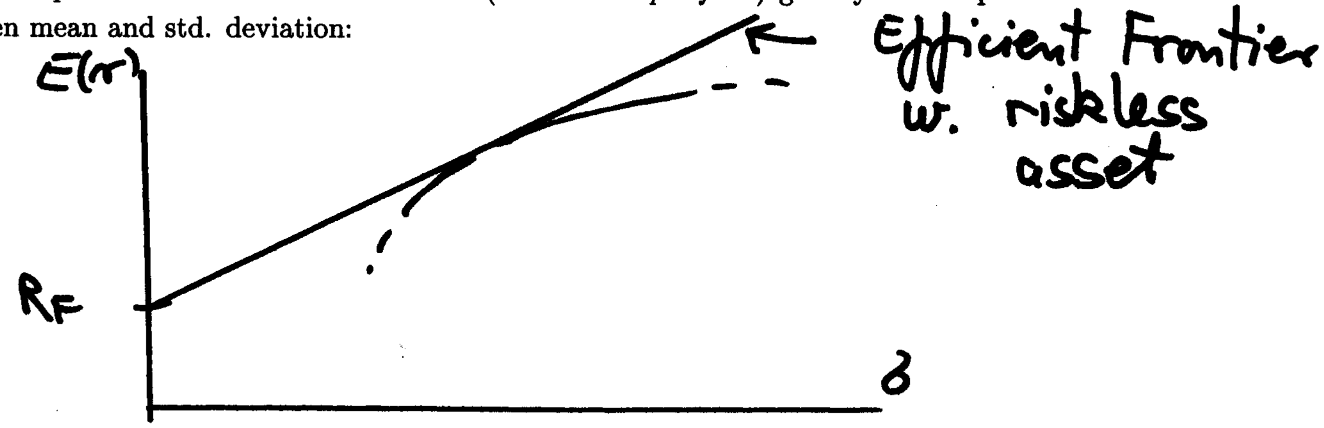
Assume you can invest in 2 assets with returns R_{1t} and R_{2t} and that $E(R_{1t}) = 1$ and $E(R_{2t}) = 5$ with $\mathrm{Var}(R_{1t}) = \mathrm{Var}(R_{2t}) = 1$ and a covariance $\mathrm{Cov}(R_{1t}, R_{2t}) = 0$. Even though asset 2 is paying a much higher return than asset 1, most investors would prefer to buy a mix of stock 1 and 2 since the diversification of risk leads to lower variance for a portfolio with both stocks relative to the undiversified portfolio of only holding stock 2. For example, if you invest 1/2 in asset 1 and 1/2 in asset 2 you get a mean return of $\mu_p = 1/2 * 1 + 1/2 * 5 = 3$. (Think of 1/2 dollar, although it could be a million dollars without changing the result. For brevity I just write 1/2 without specifying units). The variance of the half-half portfolio is $\sigma_p^2 = (1/2)^2 * 1 + (1/2)^2 * 1 = 1/2$, implying that $\sigma_p = \sqrt{\sigma_p^2} = 1/\sqrt{2} = 0.71$. You can try other portfolio weights and when you plot the portfolio mean (μ_p) against the portfolio standard deviation (σ_p), you get the mean variance frontier:



You can see from the graph that you will always choose at least 50% in asset 2, but depending on your attitude to risk, any combination with more than 50% in asset 2 may be your optimal choice. These combinations form the *efficient frontier* for these two assets. See any textbook for a more elaborate graph.

A very important thing to notice is that the shape of the mean–std. deviation frontier is convex, reflecting that diversification always improves the std. deviation better than the simple average of the standard deviations of the assets (except when assets are perfectly correlated). What if you have 3 assets? The same holds because any portfolio of 3 assets can be seen as a portfolio of 2 assets: 1 asset versus a portfolio of the other 2 assets (in mathematics this is called an induction argument). This generalizes to any number of assets so the efficient frontier will look like this:

What is the situation if investors also have access to a safe asset with return R_F? Clearly one can combine any portfolio with the safe asset. If a portfolio has mean return μ_p and std. deviation σ_p then if you consider the asset which consist of a fraction a invested in the portfolio and the rest in the safe asset, you can easily see that you get a mean return of $a*\mu_p+(1-a)*R_F = R_F+a*(\mu_p-R_F)$ and variance $a^2\sigma_p^2 + 0$ which implies a std. deviation of $a * \sigma_p$. We see that the excess return and the std. deviation both are proportional to the amount invested in the risky portfolio. One can combine the safe asset with any portfolio below the efficient frontier, but it is easy to realize that one particular portfolio on the efficient frontier (the *market portfolio*) gives you the optimal trade-off between mean and std. deviation:



The content of the above figure is that everybody will hold a combination of the market portfolio and the safe asset. From this we will derive the CAPM relation

$$E(R_{it}) = R_F + \beta_i * (E(R_{Mt}) - R_F) \ ,$$

where R_{it} is the return on asset i (at time t), R_F is the safe return, and R_{Mt} is the market return.

β_i is the covariance of R_i with R_M divided by the variance of R_M. The CAPM is a relation between an asset return, the safe rate of interest and the market return, at a point in time. When we want to estimate and test the CAPM model we need to use historical data and of course the safe return is not constant over time (this is actually subject to debate, for example, the short term treasury rate may conceivably equal a safe real rate plus a variance (expected) inflation rate . When we test the CAPM on historical data we therefore interpret the CAPM relation as a relation between the *excess returns* (the returns in excess of the safe rate) $R_{it} - R_{Ft}$ and $R_{Mt} - R_{Ft}$, where R_{Ft} is the safe return at time t. The safe return is usually measured as the return on a short term Treasury bill.

We *measure* beta from the regression

$$R_{it} - R_{Ft} = \alpha_i + \beta_i * (R_{Mt} - R_{Ft}) + e_{it} \quad (*) \ ,$$

where e_{it} is a disturbance term in the relation for stock i at time t. The CAPM is a relation between mean returns and we get from the CAPM equation to (*) by adding a disturbance term, e_{it}, which captures the deviation from the mean. Note that in this model α is 0 if the CAPM is true, so you could also estimate the restricted regression

$$R_{it} - R_{Ft} = \beta_i * (R_{Mt} - R_{Ft}) + e_{it} \quad (**) \ ,$$

and if the CAPM is true it should not matter which of (*) or (**) that you estimate. We will, however, always measure β_i from the regression (*).

To *test* if the CAPM describes the return to an *individual asset* you could make a t-test for $\alpha = 0$ in (*). To test the CAPM in general you would, however, want to test if asset returns typically satisfy the main implication of the CAPM model, namely that the variance of the e_{it}-term doesn't affect the return on the asset. (For example, if assets that are uncorrelated with the market return (β is 0) but otherwise have high variance have the same return on average as safe treasury bonds.)

An elementary derivation of the CAPM relation. Not on Exam. The following derives the CAPM relation. Unfortunately the details are not giving many economic insights.

The efficient frontier, when there is a safe asset with return R_F, consists of the portfolios that are made up of the safe asset and a unique market portfolio of risky assets. The market portfolio is the point where the line from $(0, R_F)$ is tangent to the portfolio of risky assets. We argued that this gives the optimal trade-of between mean return and risk (portfolio standard deviation). That this trade-off is optimal means that the efficient frontier is the steepest line from $(0, R_F)$ to a point (σ_p, μ_p) among the feasible portfolios.

Now note that the slope of the line from $(0, R_f)$ to an *efficient* portfolio p with mean $E(R_p) = \mu_p$ and standard deviation σ_p is

$$\text{slope from safe return :} \qquad \frac{\mu_p - R_F}{\sigma_p} \ ,$$

The return to portfolio p is made up of the fraction invested in the safe asset times the safe interest rate plus the return to the individual stocks and bonds in the market portfolio times the fraction of the portfolio p invested in each stock or bond. We will consider the return to one particular stock (GM, say) and its relation to the market portfolio, i.e., the CAPM relation. Since nothing particular is assumed about the return to GM, this proves the CAPM for any asset. We can think of the portfolio p as consisting of GM stock, the safe asset and the rest of the assets in the market. We can just consider "the rest" as an individual asset in order to focus on GM and the safe interest rate. Assume the fraction invested in GM is some number w_{GM} and the fraction invested in the safe asset is w_r. Then

$$R_p \;=\; w_{GM}R_{GM} + (1 - w_{GM} - w_r)R_F + w_rR_r \ ,$$

where R_r is the return to the portfolio minus the safe asset and the GM asset. Since the fractions has to add up to one, $1 - w_{GM} - w_r$ is the fraction invested in the safe asset.
It is easy to find that

$$\mu_p \;=\; w_{GM}E(R_{GM}) + (1 - w_{GM} - w_r)R_F + w_rE(R_r) \;\text{ and} \tag{1}$$

$$\sigma_p^2 \;=\; w_{GM}^2\text{VAR}(R_{GM}) + w_r^2\text{VAR}(R_r) + 2w_{GM}w_r\text{COV}(R_{GM}, R_r) \ . \tag{2}$$

You should ask yourself at this stage how we can be sure that GM stock is part of the market portfolio. This follows from an equilibrium argument. Everybody holds the market portfolio. If no-one wanted to by GM the price would keep on falling. At some stage the price would be so low that GM would become an attractive investment and market participants would buy GM. Or to put it differently: if there is (almost) zero demand for an asset the price will be (almost) zero, but as long as the asset pays any dividend then the price cannot be zero, so it cannot be true that there is zero demand.

The proof of the CAPM relation for the GM stock (and therefore for any stock), follows from noticing that the slope, $(\mu_p - R_F)/\sigma_p$, of the line from the pont $(0, R_F)$ to a point (μ_p, σ_p) on the efficient frontier, is a function of w_{GM}. If you increase (or decrease) the fraction, w_{GM}, invested in GM and decrease (increase) the fraction invested in the safe asset by the same amount (so it is still a feasible portfolio), the slope will decline. Of course, the slope is also a function of the fractions of other assets in the portfolio, but it turns out that we only need to consider the trade-off between R_{GM} and R_F in order to obtain the CAPM relation.

Now the mechanics. Since $[\mu_p(w_{GM}) - R_F]/\sigma_p(w_{GM})$ is the highest possible, the derivative

$$\partial(\frac{\mu_p - R_F}{\sigma_p})/\partial w_{GM} \quad ,$$

is zero (first order condition, FOC, for maximum). For simplicity, I do not write μ_p as a function of w_{GM} in the following, in order to not clutter notation.

$$\partial(\frac{\mu_p - R_F}{\sigma_p})/\partial w_{GM} = \left(\frac{\partial(\mu_p - R_F)}{\partial w_{GM}}\sigma_p - (\mu_p - R_F)\frac{\partial \sigma_p}{\partial w_{GM}}\right) \Big/ \sigma_p^2 \ ,$$

so the first order condition for maximum is

$$(*) \quad \frac{\partial \mu_p}{\partial w_{GM}} \sigma_p - (\mu_p - R_F) \frac{\partial \sigma_p}{\partial w_{GM}} = 0 \ .$$

Now notice that

$$\mu_p \ = \ w_{GM} E(R_{GM}) + (1 - w_{GM} - w_r) R_F + w_r E(R_r) \ ,$$

so

$$\partial \mu_p / \partial w_{GM} \ = \ E(R_{GM}) - R_F \ .$$

Since this is one side of the CAPM relation, it seems that we are on the right track. Differentiating σ_p^2 we get

$$\partial \sigma_p^2 / \partial w_{GM} = 2 w_{GM} \mathrm{VAR}(R_{GM}) + 2 w_r \mathrm{COV}(R_{GM}, R_r) \ \ ,$$

which you can check implies

$$\partial \sigma_p^2 / \partial w_{GM} = 2\mathrm{COV}(R_{GM}, R_p) \ \ .$$

This is promising – now the last bits to tidy up: In general, $\partial \sqrt{f(x)}/\partial x = \frac{1}{2\sqrt{f(x)}} \partial f / \partial x$, which applied to our situation (since $\sigma_p = \sqrt{\sigma_p^2}$) gives us

$$\partial \sigma_p / \partial w_{GM} = \mathrm{COV}(R_{GM}, R_p)/\sigma_p \ \ .$$

Now we collect the pieces and substitute them into the FOC (*) and get

$$[E(R_{GM}) - R_F]\sigma_p - (\mu_p - R_F)\mathrm{COV}(R_{GM}, R_p)/\sigma_p = 0 \ .$$

Now just divide by σ_p and we get

$$[E(R_{GM}) - R_F] \ = \ (\mu_p - R_F)\mathrm{COV}(R_{GM}, R_p)/\sigma_p^2 \ .$$

Since p could be any point on the Capital Market Line, the relation is also true when we choose p equal to the market portfolio, with return R_M, and we have the CAPM relation:

$$\mathrm{CAPM}: \quad [E(R_{GM}) - R_F] \ = \ (\mu_M - R_F)\mathrm{COV}(R_{GM}, R_M)/\sigma_M^2 \ ,$$

or for $\beta_{GM} = \mathrm{COV}(R_{GM}, R_M)/\sigma_M^2$:

$$\mathrm{CAPM}: \quad [E(R_{GM}) - R_F] \ = \ (\mu_M - R_F)\beta_{GM} \ ,$$

which can also be written as

$$\mathrm{CAPM}: \quad [R_{GM} - R_F] \ = \ (R_M - R_F)\beta_{GM} \ + \ u \ ,$$

where the error term u has mean 0.

Notice that for understanding (and exams) it is not the most important that you can derive the relation. The important thing is that you understand the idea (that we are finding the maximum slope of the capital market line as a function of the share invested in a given stock).