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Bent E. Sørensen

INTEREST RATE THEORY

We will cover fixed income securities. The major categories of long-term fixed income securities are federal government bonds, corporate bonds, mortgages, and municipal bonds. Government bonds can be considered risk free whereas other bonds have a risk of default. Some bonds are callable or convertible. (Fixed income securities are covered in "Asset Pricing" Cochrane, Chapter 19 and in "The Econometrics of Financial Markets" by Campbell, Lo, and MacKinlay Chapters 10 and 11.) We will not consider default risk and the pricing of callable bonds because it is better to understand the simplest case before adding the complications that arise from callability, default risk, etc.

Most longer term bonds are coupon bonds. This means that they pay a fixed amount of interest at given points in time (the **coupon payment**) and a principal payment (the **nominal value** of the bond) at maturity. The nominal value is also called the "face value" or "par value." Normally the interest payments are equi-spaced, by which I mean that they pay interest every month, half year or year. Government bonds usually pay interest semiannually. We will typically use the convention that you can buy the bond at time $t = 0$ for a price P_0 and we denote the time to maturity by T. NOTE that T is the number of periods that the bond pays interest. The cash flow to a bond will be denoted $C(1), ..., C(T)$. For a 4 year bond with a nominal value of 1000$ and annual interest payments of 50$ the cash-flows will be $C(1) = C(2) = C(3) = 50\$$ and $C(4) = 1050\$$. The coupon payment divided by the face value is called the **coupon rate**, or **current yield** and the coupon rate in the example will be .05 or 5%. The **yield to maturity** of a bond with a cash flow $C(t), t = 1, ..., T$ and a price of P_0 is the value y that solves the equation

$$P_0 \;=\; \sum_{t=1}^{T} \frac{C(t)}{(1+y)^t} \;.$$

For the 4 year bond considered above, assume that the price today is 900$. The yield to maturity will then be the y that solves

$$900 \;=\; \frac{50}{1+y} + \frac{50}{(1+y)^2} + \frac{50}{(1+y)^3} + \frac{1050}{(1+y)^4} \;.$$

You can solve this and find $y = 8.02\%$. Actually, since the equation is not linear, you may not be able to solve it unless you have a suitable calculator (or computer program). The yield correspond to a constant safe rate annual rate of interest. If the safe rate of interest was y then 1$ today could be saved and you would have $(1 + y)\$$ next year. Or to put it another way, 1$ next year would be worth $1/(1 + y)\$$ today. Similarly 50$ next year would be worth $50/(1 + y)\$$ today. And since 1$ saved for 2 years would be worth $(1 + y) * (1 + y) = (1 + y)^2$, we value 50$ in two years as

$50/(1+y)^2$\$ today. If the price of the bond today was 1000\$, rather than 900\$, you would find that the yield was 5%. In general, the yield to maturity is equal to the coupon rate if the price of the bond in the special case where the price of the bond is equal to its face value.

We will make quite heavy use of yet another yield concept, namely the **spot rate**. Spot rates are the yields to bonds that only make one payment. Such bonds are called pure discount bonds or zero-coupon bonds. They play a central role in the theory for the term structure of interest rates, but they have also lately become important in the financial market. The government usually issues discount bonds at short maturities (treasury bills) and coupon bonds at longer maturities, but many brokerage firms now buy longer term coupon bonds and sell the right to the interest payments and the right to the principal separately – this way creating "artificial" long term government discount bonds. (These principals which are stripped of the interest rate payments are known as "strips").

The one year **spot rate** is defined as the yield on a pure discount bond of one year maturity. (Spot rates are normally quoted as 2 times the six-month rate but I will assume here that interest payments are annual in order to not clutter up the notation). So for a discount bond paying X\$ in one year with a present price of P_0, you find the spot rate from solving

$$P_0 = \frac{X}{1+S_1} \Rightarrow S_1 = \frac{X}{P_0} - 1 \; .$$

For example: If $P_0 = 970.87$ and $X = 1000$, then $S_1 = 3\%$. You find the 2-year spot rate as the yield to a 2-period zero-coupon bond, i.e. for a zero-coupon bond paying off X after 2 years:

$$P_0 = \frac{X}{(1+S_2)^2} \;\; \Rightarrow \;\; S_2 = \sqrt{\frac{X}{P_0}} - 1 \; .$$

Example: If $P_0 = 950$ and $X = 1000$, then $S_2 = \sqrt{1000/950} - 1 = \sqrt{1.0526} - 1 = 2.6\%$
Similarly, for the 3-year spot rate with price P_0 and pay-off X:

$$P_0 = \frac{X}{(1+S_3)^3} \;\; \Rightarrow \;\; S_3 = (\frac{X}{P_0})^{(1/3)} - 1 \; .$$

Example: If $P_0 = 1700$ and $X = 2000$, then $S_3 = (2000/17000)^{(1/3)} - 1 = (1.1765)^{(1/3)} - 1 = 5.6\%$
In general, the T-year spot rate for a bond with price P_0 and pay-off X is

$$P_0 = \frac{X}{(1+S_T)^T} \;\; \Rightarrow \;\; S_T = (\frac{X}{P_0})^{(1/T)} - 1 \; .$$

The 1-year spot rate is usually different from the 2-year spot rate which again is different from the 3-year spot rate and so on. In other words, at any given date there is no single interest rate –

even in the absence of default risk, callability, etc. The phrase **term structure of interest rates** refers to the patterns of interest rates for safe zero-coupon bonds at any given data.

The **yield curve** is a plot of the spot rate against time to maturity – S_T on the $Y-$ axis and T on the $X-$ axis (NOTE that "T" here is NOT the date, but the time to maturity at some given time). The yield curve can take various shapes, although an upward sloping yield curve is the most common. You can find pictures of today's yield curve in publications such as the Wall Street Journal. The "term structure of interest rates" is really just another word for the "yield curve." We will study some theories about the term structure of interest rates.

We will use the notation m_t for the pricing kernel, which we will usually interpret as $\beta\frac{U'(C_t)}{U'(C_{t-1})}$. In terms of modern theory the price of a one-period discount bond at time t is $E_t\{m_{t+1}\}$ while the price of a two-period discount bond is $E_t\{m_{t+1}m_{t+2}\}$. One way to verify this is to consider that the one-year pay-out to a two-period discount bond is equivalent to selling the discount bond after a year at price $E_{t+1}\{m_{t+2}\}$ so the one period return is $E_t\{m_{t+1}E_{t+1}(m_{t+2})\} = E_tE_{t+1}\{m_{t+1}(m_{t+2})\} = E_t\{m_{t+1}m_{t+2}\}$ by the law of iterated expectation. In general the price of a T period discount bond is

$$P_0 = E_t\{m_{t+1}...m_{t+T}\}$$

We will not have time to go into modern models of the term structure but, for example, a general equilibrium model of the term structure involves building a dynamic general equilibrium model and then generating the pricing kernels from the Euler equation and then pricing the discount bonds. Of course, unless you are willing to assume something simple like a linear production technology this quickly become impossible to do analytically. See the graduate textbooks for references if you want to go into this.

An important concept when you consider the term structure of interest rates is the **forward interest rate**. The forward interest rate f_{st} is the interest rate on a loan maturing t periods from now, but with the loan taken out s periods in the future ($s < t$ of course).
If you want to lend money (buy a bond) with a two year maturity you can do this in 2 ways:

a) Buy a 2-year discount bond with 2-year return $(1 + S_2)^2$, where S_2 is the spot-rate.
or
b) Buy a 1-year discount bond with 1-year return $1 + S_1$, and write a contract that states that you are going to spend $(1 + S_1)$\$ on a bond with return f_{12}, where f_{12} is the forward interest rate for a loan signed today, to be delivered at the end of year 1 and repaid at the end of year 2.
The contracts a) and b) should have equal value, since both contracts involved delivering 1\$ (or X\$) today with a fixed return 2 years later. Therefore, the yield should be same, which means that:

$$(1 + S_2)^2 \ = \ (1 + S_1) * (1 + f_{12}) \ \ ,$$

In general, any t year loan is equivalent to an s year loan and a $t-s$ period forward loan, taken out at time s, so we have

$$(1+S_t)^t \;=\; (1+S_s)^s * (1+f_{st})^{(t-s)} \quad ,$$

which implies that

$$(1+f_{st})^{(t-s)} \;=\; \frac{(1+S_t)^t}{(1+S_s)^s}$$

or

$$f_{st} \;=\; \left(\frac{(1+S_t)^t}{(1+S_s)^s}\right)^{1/(t-s)} - 1$$

For simplicity we will just denote the one period forward rate at time s by f_s, so that $f_s = f_{s,s+1}$ by definition. We will in particular use the one period forward rates. Using the above equation we find

$$f_1 \;=\; \frac{(1+S_2)^2}{(1+S_1)} \;-\; 1 \;,$$

$$f_2 \;=\; \frac{(1+S_3)^3}{(1+S_2)^2} \;-\; 1 \;,$$

and in general

$$f_t \;=\; \frac{(1+S_{t+1})^{(t+1)}}{(1+S_t)^t} \;-\; 1 \;.$$

EXAMPLE: Assume $S_2 = 6\%$ and $S_1 = 5\%$, so the yield curve (for maturities 1 and 2 periods) is upward sloping. What is the forward rate f_1? Using the equations above we find:

$$(1+.06)^2 \;=\; (1+0.05)^1 * (1+f_1)^1 \;.$$

or

$$1+f_1 \;=\; 1.06^2/1.05 = 1.0701 \;\Rightarrow\; f_1 \;=\; .0701 \;.$$

Note that the 2-period spot rate is an "average" of the 1-period spot rate and the forward rate. The spot rate is close to, but not exactly equal to the simple arithmetic average; here (.0701+.05)/2 = .06005. Therefore, when the 2-period spot rate is 1 point above the 1-period spot rate, it implies that the forward rate is approximately 2 percentage points higher than the 1-period spot rate.
EXAMPLE: Assume $S_4 = 8\%$ and $S_2 = 7\%$. We can then find the forward rate f_{24} as

$$f_{24} \;=\; \sqrt{\frac{1.08^4}{1.07^2}} - 1 \;=\; .09 \;.$$

If you observe an upward sloping yield curve, this means that the forward rates are all higher than the spot rates, in the sense that $f_t > S_{t+1}$ for all t (remember that f_t is the interest rate on a loan delivered at t maturing in $t+1$ just as S_{t+1} is the current spot rate on a loan maturing in $t+1$). This follows since

$$1+f_t \;=\; \frac{(1+S_{t+1})^{(t+1)}}{(1+S_t)^t} \;=\; \frac{(1+S_{t+1})^t}{(1+S_t)^t} * (1+S_{t+1}) \;>\; 1+S_{t+1}\ ,$$

since

$$S_{t+1} > S_t \Rightarrow (1+S_{t+1}) > (1+S_t) \Rightarrow (1+S_{t+1})/(1+S_t) > 1 \Rightarrow \left(\frac{1+S_{t+1}}{1+S_t}\right)^t > 1\ .$$

The **pure expectations hypothesis** states that forward interest rates are unbiased predictors of corresponding future interest rates. For example, if the two-year spot rate is 6% and the one-year spot rate is 5%, then the forward rate for a 1-year loan taken out next year is (approximately) 7%. The pure expectations hypothesis states that the expected 1-year spot rate in a year is 7%. Define the one year spot rate in year t as $S_{t,1}$. The pure expectations hypothesis then can be stated as,

$$f_1 \;=\; ES_{1,1}\ ,$$

or more generally as

$$f_t \;=\; ES_{t,1}\ .$$

How to test the pure expectations hypothesis? You run the regression

$$f_t \;=\; b_0 + b_1 S_{t,1} + u_t\ ,$$

and perform an F-test for $b_0 = 0,\ b_1 = 1$. The pure expectations hypothesis is an example of a rational expectations model. It implies that investors on average can forecast the future interest rate, i.e. $f_t = S_{t,1} + u_t$, where u_t is a mean zero error term. In the literature you will often see that authors performing the "reverse regression" $S_{t,1}$ on f_t (e.g. Fama), but that does not seem to make sense. The (future) interest rates are the exogenous variables and investors forecast those subject to some errors, so if you do the reverse regression you have a "measurement error" in the regressor and as I have shown earlier, this leads to a downward bias in the estimated coefficient. Similarly, you can test if the forward rate at period 0 for a $(t-s)$-period discount bond sold in period s is the unbiased predictor for the $(t-s)$-period spot rate in period s, although testing is usually done on 1-year spot and forward rates.

In practice, you create a data set where the Y-variable is, say, the 1-year forward rates in 1960, 1961,..,1989, i.e. $\{f_{1960,1}, f_{1961,1}, .., f_{1989,1}\}$, and your X variable is the 1-year spot rates in 1961, 1962,..,1990, i.e. $\{S_{1961,1}, S_{1962,1}, .., S_{1990,1}\}$. In other words, you regress the forward rate on the spot rate that existed the following year.

The pure expectations hypothesis will be rejected if investors demand a risk premium for buying long term bonds. For example, if short term investors are risk averse, then they may require a positive risk premium (often called a **liquidity premium** in interest rate models) in order to buy long-term bonds, which they may have to sell at a random price one year later. In the example where the one-year spot rate is 5% and the two-year spot rate is 6%, it may actually be the case that investors expect that on average the one-year spot rate will be 5% the following year. So how come the forward rate is 7%? The difference, 2%, will be the risk premium. In this case we will say that the ("impure") expectations hypothesis hold if $f_t = k + ES_{t,1}$ where k is a constant – the risk premium. (Usually we simply refer to this model as the expectations hypothesis, and the **pure expectations hypothesis** will denote the situation without any risk premium). We can still test the (impure) expectations hypothesis. In the example, the risk premium for buying 2-year bonds rather than 1-year bonds is 2%. If in another year the expected future one-year spot rate is 8% then the one-year forward rate should be 8+2=10%. We test this by running the regression

$$f_t = b_0 + b_1 S_{t,1} + u_t ,$$

(the same regression as before) but now we only test if $b_1 = 1$—the coefficient b_0 will measure the risk premium. The risk premium will not necessarily be positive, in the case where long-period investors dominate the market, it will be negative, since such investors prefer long bonds and only invest in short bonds if they receive a suitable risk premium. Typically, though, we expect to find a positive risk premium. In this case the yield curve will be upward sloping, even if investors expect future interest rates to equal present interest rates on average.

In reality, our general equilibrium models are about real returns so if you consider nominal bonds the nominal return is the sum of a real return and inflation. Therefore, a large fraction of the variation in forward rates reflect expectations about future inflation. If inflation is correlated with the (real) stochastic discount factors that is one way that you might motivate risk premiums.

Forward rates and marginal utility

If we let the payout to a discount bond be 1, then the price of a k-period discount bond, in terms of marginal utility is

$$P_0^k = \beta^k \frac{E_t U'(C_{t+k})}{U'(C_t)} ,$$

so the k-period spot-rate is

$$1 + S_k = \left(\frac{U'(C_t)}{\beta^k E_t U'(C_{t+k})}\right)^{(1/k)} .$$

You can see that the rate is high if current consumption is low or if expected future consumption is high relative to current consumption. Higher (today) uncertainty about future consumption will tend to drive down interest rates via a precautionary savings motive. (Note: usually, when people say that something is caused by "a motive" it is bad writing, but here interest rates may be affected by a desire to save, not by actual saving, which may or may not be possible depending, for example, on whether goods are storable in the model.)

For forward rates, focussing on two periods, we have

$$1 + f_1 = \frac{(1 + S_2)^2}{1 + S_1} \ ,$$

so

$$1 + f_1 = \frac{U(C_t)}{\beta^2 E_t U'(C_{t+2})} / \frac{U(C_t)}{\beta E_t U'(C_{t+1})}$$

implying

$$1 + f_1 = \frac{E_t U'(C_{t+1})}{\beta E_t U'(C_{t+2})} \ .$$

Observe: a) If consumption is constant, the forward rate (and all spot rates) equals the discount rate. b) Higher expected consumption in period $t + 1$ relative to period $t + 2$ lowers the forward rate. c) Higher uncertainty about period $t + 2$ ($t + 1$) lowers (increases) the forward rate. This is as expected in terms of precautionary savings motives and relative scarcity. But the formula also clearly shows how it is the expectations at t and the relative uncertainty at t, that determines the forward rate.