ECONOMICS 7344, Spring 2020
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April 14, 2020

NOTE. Obstfeld-Rogoff (OR). Simplified notation.

Assume that agents (initially we will consider just one) live for 2 periods in an economy with uncertainty in period 2. To separate "time" from "states-of-the-world" we will refer to the random states as "A" (which occurs with probability π_A) and "B" (which occurs with probability π_B.) Later, e.g. in exams, there may be more than two states $P^s, s = 1, ...S$ and/or more than two agents and/or more than two periods but I very much agree with OR's choice: when you understand the intuition for two periods and two states-of the world, it easily carries over to large (even infinite) number of periods and states of the world.

Let the agents' endowments be

	Home			Foreign		
Period	0	1	1	0	1	1
State of the world:	n/a	A	B	n/a	A	B
Endowment ("income," "output")	Y_0	Y_1^A	Y_1^B	Y_0^*	Y_1^{*A}	Y_1^{*B}
Consumption	C_0	C_1^A	C_1^B	C_0^*	C_1^{*A}	C_1^{*B}
Probability:	1	π^A	π^B	1	π^A	π^B
Price/payout of Arrow sec. A	$\frac{p^A}{1+r}$	1	0	$\frac{p^A}{1+r}$	1	0
Price/payout of Arrow sec. B	$\frac{p^B}{1+r}$	0	1	$\frac{p^B}{1+r}$	0	1

We normalize such that $p^A + p^B$=1.

Agents optimize von Neumann-Morgenstern utility:

$$U(C_0) + \beta E U(C_1) \ ,$$

which, in the two state-of-the-world situation, is the same as

$$U(C_0) + \beta \pi^A U(C_1^A) + \beta \pi^B U(C_1^B) \ .$$

Agents optimize utility subject to the budget constraint

$$C_0 + \frac{p^A}{1+r} C_1^A + \frac{p^B}{1+r} C_1^B = Y_0 + \frac{p^A}{1+r} Y_1^A + \frac{p^B}{1+r} Y_1^B \ ,$$

where the price of period 1 consumption is normalized to unity (or, in other words, $p^A/(1+r)$ is how much period 0 consumption the agent have to surrender to get one unit of consumption in period 1, state A). [NOTE: budget constraints involve prices NOT probabilities so you always have to make that clear. In some cases, prices equal probabilities but you have to always make clear that the budget constraint is conceptually setting value of consumption equal to value of output and values are calculated with prices.]

Consider the Euler equation for the Arrow security A (it looks the same for Arrow security B with the obvious change of B for A). Its payout in state A is 1 and payout in state B is 0 so the gross return (payout divided by price) in state A is $(1+r)/p^A$ and the Euler equation becomes

$$U'(C_0) = \beta\pi^A U'(C_1^A)(1+r)/p^A \ .$$

Compare to the Euler equation for the safe asset:

$$U'(C_0) = \beta[\,\pi^A U'(C_1^A)(1+r) + \pi^B U'(C_1^B)(1+r)\,] \ .$$

What makes Arrow securities easy to work with is that the Euler equation expected value on the right hand side contains only one term. And, to remind you, your main tools are the Euler equation and the budget constraint.

Another way of writing the budget constraint is to track how much of each asset the consumer buys. Use the notation B^A and B^B for the amount of the Arrow securities purchased—if there are more than two periods you will also need a time subscript (and often we use B for the amount of the safe asset bought or sold, it doesn't matter when B can take both negative and positive values—as long as you are consistent). Then instead of writing one budget constraint, you can write down consumption in period 0 and period 1A and period 1B:

$$\begin{aligned} C_0 &= Y_0 - \frac{p^A}{\cancel{1+r}}B^A - \frac{p^B}{\cancel{1+r}}B^B \,, \\ C_1^A &= Y_1^A + B^A \,, \\ C_1^B &= Y_1^B + B^B \,. \end{aligned}$$

$\cancel{P(1A + 1B)}$

$\cancel{P(B) = \frac{p^A + p^B}{1+r}} = \frac{1}{1+r}$

$P(B) = p^A + p^B$

In this relation, we keep track of what happens in each of the future periods after you have bought Arrow securities and you can see easily how the Arrow securities can be use to "move consumption" between period 0 and the future periods and also between state A and state B (sell some Arrow securities A, for example, and buy some Arrow securities B, remembering that all trade takes place at period 0). This way of keeping track of consumption is often what you will use when I ask you to find consumption in each of the periods and states of the world, because in takes account on the budget constraint and then you just need to use the first order conditions; i.e., the Euler equations and you have two equations in two unknowns:

$$U'(\underbrace{Y_0 - \frac{p^A}{1+r}B^A - \frac{p^B}{1+r}B^B}_{C_0}) = \beta \underbrace{\pi^A}_{E} U'(\underbrace{Y_1^A + B^A}_{C_1^A}) \underbrace{\frac{1+r}{p^A}}_{R^A} \,,$$

and

$$U'(Y_0 - \frac{p^A}{1+r} B^A - \frac{p^B}{1+r} B^B) = \beta\, \pi^B\, U'(Y_1^B + B^B)\, \frac{1+r}{p^B}\,,$$

which you can solve for B^A and B^B and then plug in to get consumption.

In an economy with only a discount bond that costs $\frac{1}{1+r}$, you can also substitute for consumption and you get one Euler equation. This is the standard case where the agent can save at a non-stochastic interest rate. Let $C_0 = Y_0 - \frac{1}{1+r} B$ (B is positive or negative, so you can add or subtract B as long as you are consistent). Then $C_1^s = Y_1^s + B$ for $s = A, B$, and the Euler equation is

$$U'(Y_0 - \frac{1}{1+r} B) = \beta[\pi^A U'(Y_1^A + B)(1+r) + \pi^B U'(Y_1^B + B)(1+r)]\ .$$

This is one equation in one unknown, which you can solve for the amount of the bond and then you have consumption. (Although, it is typically harder to solve this non-linear equation than the Euler equations for Arrow securities when the agent has access to these.) Here, you see that a bond can move consumption between periods 0 and 1, but not between states-of-the-world. Also, if $\beta = \frac{1}{1+r}$ and U is quadratic then the PIH holds.

A bond (a safe asset) is an asset that pays out the same amount in each state of the world. We usually consider discount bonds that pay out the amount 1. If you buy one unit of Arrow security A and one unit of Arrow security B you will receive on unit in each state of the world. So the safe bond is equivalent to buying the two Arrow securities and people sometimes say the safe bond is a redundant asset. In fact, if there are two (S) states of the world and you have two (S) linearly independent assets, this is equivalent to having two (S) Arrow securities. (Example: an asset that pay 2 in state A and 0 in state B is not linearly independent of the Arrow security for state A: it is equivalent to buying two units the Arrow security for state A.) Two assets with the same pay-outs have to have the same prices. Note, I sometimes ask you to find the interest rate on a bond if there are not Arrow securities and then to find the interest rate on a bond if the agents has access to Arrow securities. These interest rates are typically different. The same Euler equation will hold for the bond in either case, but because the Arrow securities give the agent more options, the consumption that goes into the Euler equation will typically be different between the two cases.

If we divide the Euler equation for Arrow security A with the identical one for Arrow security B, we get:

$$1 = \frac{\pi^A U'(C_1^A)/p^A}{\pi^B U'(C_1^B)/p^B}\ ,$$

or

$$\frac{p^A}{p^B} = \frac{\pi^A U'(C_1^A)}{\pi^B U'(C_1^B)}\ ,$$

or

$$\frac{p^A}{p^B}\frac{\pi^B}{\pi^A} = \frac{U'(C_1^A)}{U'(C_1^B)} \; .$$

Assuming (as we always do) the utility functions are strictly concave, then $U'(C_1^A) = U'(C_1^B)$ if and only if $C_1^A = C_1^B$. We, therefore, have $C_1^A = C_1^B$ if and only if $\frac{p^A}{p^B}\frac{\pi^B}{\pi^A} = 1$ which happens if and only if $p^A = \pi^A$ (if you need to convince yourself, use that $p^B = (1-p^A)$ and $\pi^B = (1-\pi^A)$). The case where prices equal probabilities (or more generally are proportional to) is denoted the case of "actuarially fair prices." So the consumer will choose to eliminate consumption uncertainty (by trading in the Arrow securities) if and only if prices are actuarially fair. So, even though consumer, everything else equal, prefers to avoid uncertainty the consumer will only do so if it is not too expensive; for example, if the consumer has lower output in state A than state B will consumer will not buy Arrow securities to make up for this, if the state A Arrow securities are too expensive. This is just like you won't buy, say, flood insurance if it is too expensive.

It is often easiest to impose the budget constraint by using the symbol for how much of the asset the agent purchase. To see what I have in mind consider at "bonds only" economy. (This is the standard case where the agent can save at a non-stochastic interest rate.) Let $C_0 = Y_0 - B$ (B is positive or negative, so you can add or subtract B as long as you are consistent). Then $C_1^s = Y_1^s + B(1+r)$ and the Euler equation is

$$U'(Y_0 - B) = \beta[\pi^A U'(Y_1^A + B(1+r))(1+r) + \pi^B U'(Y_1^B + B(1+r))(1+r)] \; .$$

Comments: make sure to understand why $(1+r)$ occurs where it does. Also, if $\beta = \frac{1}{1+r}$ and U is quadratic then the PIH holds.

General Equilibrium

Consider the situation where two agents make up the world (in exams I occasionally consider 3 agents or, less often, N agents, but the intuition is best learned from the 2-agents case). We really have in mind a situation with a large number of agents of each type (wherefore agents are price takes, who takes prices as given) but modeling only two agents simplifies notation a lot. The book is written in the language of international macro and I will follow that here and call the second agent "foreign" although in exams and home works I sometimes refer to, e.g., Smith and Jones, because the theory is general and not limited to international settings (historically, the applications in international came later). So, in the notation of the book, the foreign agent has variables marked with *, for example, consumption in state s time t is C_t^{*s} and endowment is Y_t^{*s} (prices are of course the same for all agents since they trade with each other, don't *ever* use different prices for different agents).

To find prices, we need to use the adding-up constraint that total consumption equals total output. There is absolutely no way to find prices without using the adding up condition, prices are found to clear the markets; i.e., to equate demand and supply or, in other words, to equate consumption (demand) to output (supply, which is exogenous here). We also need to use the Euler equations which captures the agents' optimizing behavior so we have (for Arrow security "A")

$$\begin{aligned} U'(C_0) &= \beta \pi^A U'(C_1^A) \frac{1+r}{p^A} \\ U'(C_0^*) &= \beta \pi^A U'(C_1^{A*}) \frac{1+r}{p^A} \end{aligned}$$

In order to use the adding-up constraint, that at each time t for each state s $C_t^s + C_t^{s*} = Y_t^{Ws}$ where W denotes "world" output, we have to be able to invert U'. We can do this for utility functions of the CRRA (including logarithmic), exponential, and quadratic type. Here, I follow the book and do the derivation for CRRA (the most used in applications) but you are expected to be able to find prices and interest rates also for quadratic and exponential.

For CRRA, $U(C) = \frac{1}{1-\rho} C^{1-\rho}$ where ρ is coefficient of risk aversion (sometimes we use other symbols, and ρ may be the discount factor, I can't help that, people use the symbols differently). Marginal utility for CRRA is $U'(C) = C^{-\rho}$ and we have

$$\begin{aligned} C_0^{-\rho} &= \beta \pi^A (C_1^A)^{-\rho} \frac{1+r}{p^A} \\ (C_0^*)^{-\rho} &= \beta \pi^A (C_1^{A*})^{-\rho} \frac{1+r}{p^A} \ . \end{aligned}$$

[I may write $C_1^{A-\rho}$ rather than the more precise, but cumbersome, $(C_1^A)^{-\rho}$]. Now invert U' by taking both sides of the equality signs to the power $\frac{-1}{\rho}$ and reorder a bit and we have

$$C_0 = C_1^A (\beta \pi^A \frac{1+r}{p^A})^{\frac{-1}{\rho}} \tag{1}$$

$$C_0^* = C_1^{A*} (\beta \pi^A \frac{1+r}{p^A})^{\frac{-1}{\rho}} \ , \tag{2}$$

and we can now add the equations and use the adding-up constraints to get

$$Y_0^W = Y_1^{AW} (\beta \pi^A \frac{1+r}{p^A})^{\frac{-1}{\rho}} \ .$$

By taking each side to the power $-\rho$ and reordering to get the price on the left-hand side, we get

$$\frac{p^A}{1+r} = \beta \pi^A \frac{Y_1^{AW-\rho}}{Y_0^{W-\rho}} \ .$$

The price for Arrow security "B" is of course similar, substituting B for A everywhere. You should observe (and appreciate the intuition—I will be asking for it):

- The price of the Arrow security is proportional to the probability. Agents are willing ("want") to pay more for something that is more likely to pay out. (It is like lottery tickets, you pay twice as much for two tickets, i.e., you pay twice as much to double the probability.)

- The price is proportional to the discount factor β. Keep in mind that by buying a unit of the Arrow security "you" pay in period 0 and receive something back (if any) in period 1, so if you discount the future relatively less (higher β) then the period 1 good is worth relatively more in terms of period 0 goods.

- Prices reflect (world) relative scarcity. If world output is high in state A (whether if occurs or not) the price of the Arrow security is lower. If output is high in period 0 the price of the Arrow security is higher (we are willing to give up more period 0 goods).

- (Really a sub-point of the previous bullet point) How strongly relative scarcity affects prices depends on how much higher agents marginal utility is in the low-output state compared to the high-output state: relative scarcity matters more for prices when the curvature of the utility function is high; i.e., when ρ is large.

The interest rate is now easily found as $\frac{1}{1+r} = \frac{p^A}{1+r} + \frac{p^B}{1+r}$ so

$$\frac{1}{1+r} = \beta\,(\pi^A \frac{Y_1^{AW-\rho}}{Y_0^{W-\rho}} + \pi^B \frac{Y_1^{BW-\rho}}{Y_0^{W-\rho}})$$

or

$$1 + r = \frac{1}{\beta} \frac{Y_0^{W-\rho}}{\pi^A Y_1^{AW-\rho} + \pi^B Y_1^{BW-\rho}}$$

which we can also write as

$$1 + r = \frac{1}{\beta} \frac{Y_0^{W-\rho}}{E_0\{Y_1^{W-\rho}\}}$$

Observe:

- the interest rate is inversely proportional to the discount factor β. Buying a discount bond means giving up $\frac{1}{1+r}$ in period 0 and receiving one unit in period 1, so if you discount the future relatively less (higher β) then the period 1 good is worth relatively more in terms of period 0 goods, so the interest rate is lower.

- Prices reflect (world) relative scarcity. If world output is high in period 1 (on average, but to bring home the point consider the situation were there is no world uncertainty; i.e., $Y_1^{AW} = Y_1^{BW}$) the price of a discount bond is lower or, equivalently, the interest rate is higher. (You can think of the interest rate as the amount "period 1" is willing to compensate "period 0" to postpone consumption, if there is a large amount of output in period 1, this will be high because marginal utilities are low when output is abundant.) If output is high in period 0, the interest rate is lower—people want to save some of the high output but they can't in a no-storage economy, so the interest will fall low enough to discourage saving (i.e., until desired world saving is 0).

- As we found for the Arrow-securities, relative scarcity has a stronger effect when the curvature of the utility function is high.

- The denominator is the expectation of a convex function so by Jensen's inequality, it is larger than if output was certain with the same mean. In other words, if there is uncertainty the interest rate is lower. This is because people desire to save more (precautionary saving) but since the world can't save, the interest will fall (compared to the no uncertainty situation) until desired world saving is 0. In general, the more uncertainty the lower the interest rate. (We can't always compare situations in terms of more or less uncertainty but in exam questions it should be obvious, like uncertainty versus no uncertainty, or if, say, $\pi^A = \pi^B = 0.5$ then $Y_1^{AW} = 10$ and $Y_1^{BW} = 20$ is clearly more uncertain than $Y_1^{AW} = 14$ and $Y_1^{BW} = 16$ (the mean is 15 in either situation).

An important implication of the Euler equations (1) and (2) for home and foreign is (take the ratio of (1) to (2))

$$\frac{C_0}{C_0^*} = \frac{C_1^s}{C_1^{s*}}$$

for all states of the world. If the ratio of home to foreign consumption is always constant that implies that

$$C_t^s = k\, Y_t^{Ws} \tag{3}$$

for all s and t. In applied work, the focus is usually on the implication

$$\Delta \log(C_t) = \Delta \log(C_t^W) \ .$$

(This is for all states of the world but in a sample of data we, of course, only observe one realization and we rarely use the s superscript.)

To find k use the budget constraint with the expression from (3) substituted in (using Y^W rather than C^W to highlight that world consumption is exogenous):

$$k\, Y_0^W + \frac{p^A}{1+r} k\, Y_1^{WA} + \frac{p^B}{1+r} k Y_1^{BW} = Y_0 + \frac{p^A}{1+r} Y_1^A + \frac{p^A}{1+r} Y_1^B \ ,$$

from which k is easily found as

$$k = \frac{Y_0 + \frac{p^A}{1+r} Y_1^A + \frac{p^A}{1+r} Y_1^B}{Y_0^W + \frac{p^A}{1+r} Y_1^{WA} + \frac{p^B}{1+r} Y_1^{BW}} \ .$$

We see that k is value of home's output divided by the value of world output—and, of course, that is what is has to be because it is a market outcome.