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1 The effect of income shocks on consumption in Hall's '78 model

This note basically summarizes pp.81–87 of Deaton's (1992) book "Understanding Consumption" (with an attempt to spell out some issues in more detail).

The goal here is to predict the impact of a "shock to income" on permanent income. A "shock to income" is jargon for the difference between actual income at period t and the expected value of period t income where the expectations are those of period $t-1$. This shock will also affect the expectations of future income on which the PIH consumer bases their consumption decision at period t. So we need to find $E_t y_{t+k}$ for all k (the period-t expected income in period $t+k$) and compare it to our previous expectations of y_{t+k}, that is $E_{t-1} y_{t+k}$. The change in consumption in the PIH (and any other rational expectations forward looking model) will depend on the changes in these expectations.

Assume that income follows a stationary invertible ARMA time series model. First note that if income y_t follows a (maybe infinite) invertible MA-model,

$$y_t = \mu + u_t + b_1 u_{t-1} + b_2 u_{t-2} + ...$$

then the shock to income is $u_t = y_t - E_{t-1} y_t$. In other words, we have the intuitive observation that

$$y_t = E_{t-1} y_t + u_t \ ,$$

i.e., what "we expected" y_t would be plus the innovation. This is of course why we call u_t an innovation. We can *always* define the innovation like this in any (linear or non-linear) time series. But in ARMA models, all *new* information about all future values is a function of u_t as we shall see.

Note that we also can write the conditional expectation as

$$E_{t-1}(y_t) \equiv E(y_t | y_{t-1}, y_{t-2}, ...) = E(y_t | u_{t-1}, u_{t-2}, ...) \ ,$$

in this case where y_t only depends on its own lagged values. This is because for any stationary invertible ARMA process, we can move back and forth between the AR and the MA models:

$$a(L)x_t = \mu + b(L)u_t \quad \Leftrightarrow \quad x_t = \frac{\mu}{a(1)} + a^{-1}(L)b(L)u_t \quad \Leftrightarrow \quad b^{-1}(L)a(L)x_t = \frac{\mu}{b(1)} + u_t \ ,$$

so, when y_t is stationary and invertible, the y_t's and the u_t's can be derived from each other.

Now, because all error terms (innovations, sometime the name slips, because in estimations it is usually an error term) at or before $t-1$ are known at $t-1$ while the period t innovation has mean 0, we have

$$E_{t-1}(y_t) = \mu + b_1 u_{t-1} + b_2 u_{t-2} + ...$$

which implies

$$E_t(y_{t+1}) = \mu + b_1 u_t + b_2 u_{t-1} + ...$$

and similarly

$$E_{t-1}(y_{t+1}) = \mu + b_2 u_{t-1} + b_3 u_{t-2} + ...$$

Note. I hope you are all familiar with the **Law of Iterated Expectations** which states that to take the expectation of a random variable y_{t+1} with respect to an information set (say, I_{t-2}), you can take the expectation with respect to a larger information set (say, I_{t-1}) and then take the expectation of that expression with respect to a smaller information set I_{t-2}. This is used a lot in macro and elsewhere. So we have, for example

$$E_{t-2}y_{t+1} = E_{t-2}\{E_{t-1}y_{t+1}\},$$

because the information set at $t-1$ is larger (u_{t-1} is the extra information) than the information set at $t-2$. The unconditional expectation "conditions on nothing," so for example

$$Ey_{t+1} = E\{E_{t-1}y_{t+1}\}.$$

The basic intuition when you take conditional expectation like this is simply that one can consider u_s for all s before the "current" time period (e.g., t or $t-1$) as known.

Continuing, we have

$$E_t(y_{t+2}) = \mu + b_2u_t + b_3u_{t-1} + ...$$

and

$$E_{t-1}(y_{t+2}) = \mu + b_3u_{t-1} + b_4u_{t-2} + ...$$

The pattern is now obvious, and we see that $y_t - E_{t-1}y_t = u_t$, $E_ty_{t+1} - E_{t-1}y_{t+1} = b_1u_t$,

$E_t y_{t+2} - E_{t-1} y_{t+2} = b_2 u_t \ldots.$, so that all new information on future expected income is a function of the present innovation u_t.

A maybe simpler, equivalent way to arrive at this conclusion is to observe that when $y_t = \mu + u_t + b_1 u_{t-1} + b_2 u_{t-2} + ..$ then $\partial y_t / \partial u_t = 1$, $\partial y_t / \partial u_{t-1} = b_1$,$\partial y_t / \partial u_{t-2} = b_2 \ldots$ and therefore also $\partial y_t / \partial u_t = 1$, $\partial y_{t+1} / \partial u_t = b_1$,$\partial y_{t+2} / \partial u_t = b_2 \ldots.$ Since, at any period $t + s$ where $s \geq 0$ the expectation at time t of $u_{t+s} = 0$ and u_s where $s \leq t$ are known at time t as well as at time $t - 1$ the change in the expected value of future income is given as the partial derivative of those future income wrt. u_t times the value of u_t.

A plot of b_k against k is called an <u>impulse response function</u> because it measures the response of future income to the innovation or "impulse" u_t.

Now return to Hall's version of the PIH. Hall's model implies that $c_t = E_t c_{t+1}$. Assume that this relation holds in all future periods and that the time horizon is infinite. Then the <u>budget constraint</u> is

$$\sum_{k=0}^{\infty} (1+r)^{-k} c_{t+k} = A_t + \sum_{k=0}^{\infty} (1+r)^{-k} y_{t+k}$$

which implies

$$\sum_{k=0}^{\infty}(1+r)^{-k}E_t c_{t+k} = A_t + \sum_{k=0}^{\infty}(1+r)^{-k}E_t y_{t+k}$$

because the martingale condition holds in all future periods we have $E_t c_{t+k} = c_t$ for all $k \geq 0$ (by the "law of iterated expectations") and the left hand side of the displayed equation becomes $\sum_{k=0}^{\infty}(1+r)^{-k}c_t = c_t\,(1+r)/r$ (remember—and you have to remember that one—that $1 + a + a^2 + ... = \frac{1}{1-a}$) . When you plug in $\frac{1}{1+r}$ for a, you get $\frac{1+r}{r}$ after multiplying numerator and denominator by $1+r$.

$$(1) \qquad \frac{1+r}{r}c_t = A_t + \sum_{k=0}^{\infty}(1+r)^{-k}E_t y_{t+k} \ ,$$

or

$$(2) \qquad c_t = \frac{r}{1+r}A_t + \frac{r}{1+r}\sum_{k=0}^{\infty}(1+r)^{-k}E_t y_{t+k} \ .$$

which implies (as the variables are stationary)

$$(3) \qquad c_{t-1} = \frac{r}{1+r}A_{t-1} + \frac{r}{1+r}\sum_{k=0}^{\infty}(1+r)^{-k}E_{t-1} y_{t-1+k} \ .$$

We want to find Δc_t, so we need to line up the future income shocks; that is, write the last summation in terms of y_{t+k} not y_{t-1+k}. We have

$$(4) \qquad \sum_{k=0}^{\infty}(1+r)^{-k}E_t y_{t+k} = y_t + (1+r)^{-1}E_t y_{t+1} + (1+r)^{-2}E_t y_{t+2} + ...$$

We can also change t to $t-1$ here:

$$\begin{aligned}\sum_{k=0}^{\infty}(1+r)^{-k}E_{t-1}y_{t-1+k} &= y_{t-1}+(1+r)^{-1}E_{t-1}y_t+(1+r)^{-2}E_{t-1}y_{t+1}+(1+r)^{-3}E_{t-1}y_{t+2}\ldots. \\ &= y_{t-1}+(1+r)^{-1}(E_{t-1}y_t+(1+r)^{-1}E_{t-1}y_{t+1}+(1+r)^{-2}E_{t-1}y_{t+2}\ldots.)\end{aligned}$$

So that

$$\sum_{k=0}^{\infty}(1+r)^{-k}E_{t-1}y_{t-1+k} = y_{t-1}+(1+r)^{-1}\sum_{k=0}^{\infty}(1+r)^{-k}E_{t-1}y_{t+k}\,,$$

where the latter summation contains the same future y's as for c_t so it is easy to subtract terms.

Keep in mind where we are going. We want to find $\Delta c_t = c_t - c_{t-1}$. c_t contains a sum of terms in y_{t+k}, and c_{t-1} contains a sum of terms in y_{t-1+k}...but these are the same terms differently labelled, except for y_{t-1} which is not in the expression for c_t. So we separate that out.

Multiplying the expression for c_{t-1} with $(1+r)$, we get

$$(5)\qquad (1+r)c_{t-1} = rA_{t-1}+ry_{t-1}+\frac{r}{1+r}\sum_{k=0}^{\infty}(1+r)^{-k}E_{t-1}y_{t+k}\ ,$$

where the second part of the expression is very similar to the term in c_t. Formula (**??**)

contains lagged assets, so to line up current and lagged consumption we use the dynamic budget constraint: $A_t = (A_{t-1} + y_{t-1} - c_{t-1}) * (1+r)$ that y and c takes place at the beginning of the period an the interest will be on assets left-over after consumption. Equation (1) then implies

$$(6) \qquad c_t = r(A_{t-1} + y_{t-1} - c_{t-1}) + \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} E_t y_{t+k} \ .$$

We can rewrite (5) as

$$(7) \quad c_{t-1} = rA_{t-1} + ry_{t-1} - rc_{t-1} + \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} E_{t-1} y_{t+k} \ ,$$

Subtract (7) from (6) and get

$$\Delta c_t = \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} (E_t - E_{t-1}) y_{t+k} \ ,$$

(where, for any stochastic variable, $(E_t - E_{t-1})x_{t+k} \equiv E_t x_{t+k} - E_{t-1} x_{t+k}$).

Now assume that y_t follows an (possibly infinite) MA model as above. Then

$$\Delta c_t = \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} b_k u_t \ .$$

If we use $b(L)$ to denote the lag-polynomial $b(L) = 1 + b_1 L + b_2 L^2 + ...$ and $b(z)$ to denote the corresponding z-transform, then

$$\Delta c_t = \frac{r}{1+r} u_t \ \times \ (1 + b_1 \frac{1}{1+r} + b_2 (\frac{1}{1+r})^2 + b_3 (\frac{1}{1+r})^3 + ...) = \frac{r}{1+r} u_t \ \times \ b(\frac{1}{1+r}) \ .$$

What a beautiful compact formula for how consumption changes as function of a change in the expected income in all future periods. But it gets better, much better: A general ARMA process $a(L)y_t = b(L)u_t$ is equal to the infinite MA model $y_t = a(L)^{-1}b(L)u_t$, so for a general ARMA process we obtain

$$\Delta c_t = \frac{r}{1+r} u_t \times \frac{b(\frac{1}{1+r})}{a(\frac{1}{1+r})} \ .$$

This is much better because we work more often with AR model than with MA model. But it you want to prove the formula without first going the MA-representation, well, good luck with that! (Which is American vernacular for "you will never get through with that.")

NOTE: This formula is valid as long as $a(\frac{1}{1+r})$ takes a finite value. It is not actually necessary that the AR-part is stable when r is positive as the powers in $\frac{1}{1+r}$ drives down the coefficients in the infinite sum.

1.1 Excess Smoothness

Macroeconomic data for aggregate income is well approximated by an AR(1) model in differences:

$$\Delta y_t = \mu + a\Delta y_{t-1} + u_t \ ,$$

where $a > 0$, and typically $0 < a < .6$ or so. Some researchers find a significant coefficient to twice lagged income, but that coefficient is almost always found to be small and the quantitative conclusions of the following will hold for that model also. We will, therefore, illustrate the issue using the simple AR(1) model for differenced income.

The model for income can also be written as

$$(1-L)(1-aL)y_t = u_t \ ,$$

or

$$a(L)y_t = u_t \ \text{ for } \ a(L) = (1-L)(1-aL) = 1-(1+a)L + aL^2 \ .$$

Applying equation (1) to predict the change in consumption in this case gives us

$$\Delta c_t = u_t \frac{r}{1+r} \times \frac{1}{1 - \frac{1+a}{1+r} + \frac{a}{(1+r)^2}} \ ,$$

which simplifies to

$$\Delta c_t = u_t \frac{1+r}{1+r-a} \ .$$

This formula reveals that Δc_t reacts more than one-to-one with innovations to income when a is positive. This is a surprising implication of the PIH, which historically was suggested as an explanation of why consumption "is more smooth than income," and it is occasionally referred to as "Deaton's paradox".

Another way of looking at this is to consider the coefficient to income in a regression of (differenced) consumption on (differenced) income. As previously mentioned the coefficient will (for the number of observations becoming infinite) be

$$\frac{cov(\Delta c_t, \Delta y_t)}{var(\Delta y_t)} \;=\; \frac{1+r}{1+r-a} / \frac{1}{1-a^2} = \frac{1+r-a^2 * (1+r)}{1+r-a} \;,$$

which is larger than one for typical values of a and r. One way of testing the PIH is to regress differenced consumption on differenced income and see if the coefficient is equal to that predicted by the PIH or — at the least — larger than one, but that is usually **not** done when using macroeconomic data because income may not be a valid regressor. (Technically, an innovation to consumption due to, say, a change in consumer confidence, may change the level of income (as in the IS/LM model) making income partly a function of consumption. In the language of econometricians income is not necessarily exogenous for consumption.)

Due to these technical issues, some researchers (in particular, Deaton, who brought up the issue) have simply compared the variance of consumption changes to the variance of innovations to income. Contrary to the implications of the PIH, the latter has been found to be clearly larger than the former, and this results has become known as the "excess smoothness of consumption."