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1 The effect of income shocks on consumption in Hall's '78 model

This note basically summarizes pp.81–87 of Deaton's (1992) book "Understanding Consumption" (with an attempt to spell out some issues in more detail).
The goal here is to predict the impact of a "shock to income" on permanent income. A "shock to income" is jargon for the difference between actual income at period t and the expected value of period t income where the expectations are those of period $t-1$.

Assume that income follows a stationary invertible ARMA time series model. First note that if income y_t follows a (maybe infinite) invertible MA-model,

$$y_t = \mu + u_t + b_1 u_{t-1} + b_2 u_{t-2} + ...$$

then the shock to income is $y_t - E_{t-1} y_t$, where the conditional expectation more precisely is

$$E_{t-1}(y_t) \equiv E(y_t | y_{t-1}, y_{t-2}, ...) = E(y_t | u_{t-1}, u_{t-2}, ...) \ ,$$

where the $\equiv$ sign follows since y_t is stationary and invertible so that the y_t's and the u_t's can be derived from each other. Now

$$E_{t-1}(y_t) = b_1 u_{t-1} + b_2 u_{t-2} + ...$$

which implies

$$E_t(y_{t+1}) = b_1 u_t + b_2 u_{t-1} + ...$$

and similarly

$$E_{t-1}(y_{t+1}) = b_2 u_{t-1} + b_3 u_{t-2} + ...$$

The basic intuition is simply that one can consider u_s for all s before the "current" time period (e.g., t or $t-1$) as known. Continuing, we have

$$E_t(y_{t+2}) = b_2 u_t + b_3 u_{t-1} + ...$$

and

$$E_{t-1}(y_{t+2}) = b_3 u_{t-1} + b_4 u_{t-2} + ...$$

The pattern is now obvious, and we see that $y_t - E_{t-1} y_t = u_t$, $E_t y_{t+1} - E_{t-1} y_{t+1} = b_1 u_t$, $E_t y_{t+2} - E_{t-1} y_{t+2} = b_2 u_t ....$, so that all new information on future expected income is a function of the present innovation u_t.

A maybe simpler, equivalent way to arrive at this conclusion is to observe that when $y_t = u_t + b_1 u_{t-1} + b_2 u_{t-2} + ..$ then $\partial y_t / \partial u_t = 1$, $\partial y_t / \partial u_{t-1} = b_1$,$\partial y_t / \partial u_{t-2} = b_2$ ... and therefore also $\partial y_t / \partial u_t = 1$, $\partial y_{t+1} / \partial u_t = b_1$,$\partial y_{t+2} / \partial u_t = b_2$ .... Since, at any period $t+s$ where $s \geq 0$ the expectation at time t of $u_{t+s} = 0$ and u_s where $s \leq t$ are known at time t as well as at time $t-1$ the change in the expected value of future income is given as the partial derivative of those future income wrt. u_t times the value of u_t.

A plot of b_k against k is called an impulse response function since it measures the response of future income to the innovation or "impulse" u_t.

Now return to Hall's version of the PIH. Hall's model implies that $c_t = E_t c_{t+1}$. Assume that this relation holds in all future periods and that the time horizon is infinite. Then the budget constraint is

$$\sum_{k=0}^{\infty} (1+r)^{-k} c_{t+k} = A_t + \sum_{k=0}^{\infty} (1+r)^{-k} y_{t+k}$$

which implies

$$\sum_{k=0}^{\infty} (1+r)^{-k} E_t c_{t+k} = A_t + \sum_{k=0}^{\infty} (1+r)^{-k} E_t y_{t+k}$$

since the martingale condition holds in all future periods we have $E_t c_{t+k} = c_t$ for all $k \leq 0$ (by the "law of iterated expectations" and the left hand side of the displayed equation becomes $\sum_{k=0}^{\infty} (1+r)^{-k} c_t = c_t \, (1+r)/r$ from which

$$(1) \qquad c_t = \frac{r}{1+r} A_t + \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} E_t y_{t+k} \ .$$

Equation (1) implies (by "moving the index one period back" and taking y_{t-1} "out of the summation")

$$(2) \qquad (1+r) c_{t-1} = r A_{t-1} + r y_{t-1} + \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} E_{t-1} y_{t+k} \ .$$

(I encourage you to fill in all the steps involved in going from (1) to (2), it is not hard, but it is easy to get it messed up.) Equation (1) also implies

$$(3) \qquad c_t = r(A_{t-1} + y_{t-1} - c_{t-1}) + \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} E_t y_{t+k} \ .$$

Subtract (2) from (3) and get

$$\Delta c_t = \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} (E_t - E_{t-1}) y_{t+k} \ ,$$

(where, for any stochastic variable, $(E_t - E_{t-1})x_{t+k} \equiv E_t x_{t+k} - E_{t-1}x_{t+k}$).

Now assume that y_t follows an (possibly infinite) MA model as above. Then

$$\Delta c_t = \frac{r}{1+r} \sum_{k=0}^{\infty} (1+r)^{-k} b_k u_t \ .$$

If we use $b(L)$ to denote the lag-polynomial $b(L) = 1 + b_1 L + b_2 L^2 + ...$ and $b(z)$ to denote the corresponding z-transform, then

$$\Delta c_t = \frac{r}{1+r} u_t \times (1 + b_1 \frac{1}{1+r} + b_2 (\frac{1}{1+r})^2 + b_3 (\frac{1}{1+r})^3 + ...) = \frac{r}{1+r} u_t \times b(\frac{1}{1+r}) \ .$$

A general ARMA process $a(L)y_t = b(L)u_t$ is equal to the infinite MA model $y_t = a(L)^{-1}b(L)u_t$, so for a general ARMA process we obtain

$$\Delta c_t = \frac{r}{1+r} u_t \times \frac{b(\frac{1}{1+r})}{a(\frac{1}{1+r})} \ .$$

NOTE: This formula is valid as long as the $b-$ polynomial in invertible and the $a(\frac{1}{1+r})$ takes a finite value. It is not actually necessary that the AR-part is stable.

1.1 Excess Smoothness

Macroeconomic data for aggregate income is well approximated by an AR(1) model in differences:

$$\Delta y_t = \mu + a \Delta y_{t-1} + u_t \ ,$$

where $a > 0$, and typically $0 < a < .6$ or so. Some researchers find a significant coefficient to twice lagged income, but that coefficient is almost always found to be small and the quantitative conclusions of the following will hold for that model also. We will, therefore, illustrate the issue using the simple AR(1) model for differenced income.
The model for income can also be written as

$$(1 - L)(1 - aL)y_t = u_t \ ,$$

or

$$a(L)y_t = u_t \ \text{ for } \ a(L) = (1 - L)(1 - aL) = 1 - (1 + a)L + aL^2 \ .$$

Applying equation (1) to predict the change in consumption in this case gives us

$$\Delta c_t = u_t \frac{r}{1+r} \times \frac{1}{1 - \frac{1+a}{1+r} + \frac{a}{(1+r)^2}} \ ,$$

which simplifies to

$$\Delta c_t = u_t \, \frac{1+r}{1+r-a} \ .$$

This formula reveals that Δc_t reacts more than one-to-one with innovations to income when a is positive. This is a surprising implication of the PIH, which historically was suggested as an explanation of why consumption "is more smooth than income," and it is occasionally referred to as "Deaton's paradox".

Another way of looking at this is to consider the coefficient to income in a regression of (differenced) consumption on (differenced) income. As previously mentioned the coefficient will (for the number of observations becoming infinite) be

$$\frac{cov(\Delta c_t, \Delta y_t)}{var(\Delta y_t)} \ = \ \frac{1+r}{1+r-a} / \frac{1}{1-a^2} = \frac{1+r-a^2*(1+r)}{1+r-a} \ ,$$

which is larger than one for typical values of a and r. One way of testing the PIH is to regress differenced consumption on differenced income and see if the coefficient is equal to that predicted by the PIH or — at the least — larger than one, but that is usually **not** done when using macroeconomic data since income may not be a valid regressor. (Technically, an innovation to consumption due to, say, a change in consumer confidence, may change the level of income (as in the IS/LM model) making income partly a function of consumption. In the language of econometricians income is not necessarily exogenous for consumption.)

Due to these technical issues, some researchers (in particular, Deaton, who brought up the issue) have simply compared the variance of consumption changes to the variance of innovations to income. Contrary to the implications of the PIH, the latter has been found to be clearly larger than the former, and this results has become known as the "excess smoothness of consumption."