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# SAMPLE QUESTIONS OF MATHEMATICS 2414

Three hours are allotted for this examination: 1 hour and 30 minutes for Section I, which consists of multiple-choice questions, and 1 hour and 30 minutes for Section II, which consists of problems requiring written solutions.

## Section I. Multiple-Choice Questions

Section I consists of 45 questions. In this section of the examination, as a correction for haphazard guessing, one-fourth of the number of questions answered incorrectly will be subtracted from the number of questions answered correctly. Following are the directions for Section I and a representative set of 24 questions. (Answers are given at the end.)

*Directions:* Solve each of the following problems, using the available space for scratch-work. Then decide which is the best of the choices given and fill in the corresponding oval on the answer sheet. No credit will be given for anything written in the examination booklet. Do not spend too much time on any one problem.

*Notes:* (1) In this examination, *ln x* denotes the natural logarithm of *x* (that is, logarithm to the base *e*.) (2) Unless otherwise specified, the domain of a function *f* is assumed to be the set of all real numbers x for which *f (x)* is a real number.

1. In decomposing $\dfrac{3x^2-2x+19}{(x-2)(x^2+5)}$ by the method of partial fractions, one of the fractions obtained is

   (A) $\dfrac{2}{x-2}$ (B) $\dfrac{3}{x-2}$ (C) $\dfrac{4}{x-2}$ (D) $\dfrac{9}{x-2}$ (E) $\dfrac{27}{x-2}$

2. The normal to the curve represented by the equation $y = x^2 + 6x + 4$ at the point (-2, -4) also intersects the curve at $x =$

   (A) $-6$ (B) $-\dfrac{9}{2}$ (C) $-\dfrac{7}{2}$ (D) $-3$ (E) $-\dfrac{1}{2}$

3. If $y = x + \sin(xy)$, then $\dfrac{dy}{dx} =$

   (A) $1 + \cos(xy)$
   (B) $1 + y\cos(xy)$
   (C) $\dfrac{1}{1-\cos(xy)}$
   (D) $\dfrac{1}{1-x\cos(xy)}$
   (E) $\dfrac{1+y\cos(xy)}{1-x\cos(xy)}$

4. The graph in the $xy$-plane represented by $x=\sec t$ and $y=\tan t$, for $-\dfrac{\pi}{2}<t<\dfrac{\pi}{2}$, is

   (A) a circle (B) a semicircle (C) an ellipse (D) a hyperbola

5. What are all values of $x$ for which the series $\sum_{n=1}^{\infty}\dfrac{n3^n}{x^n}$ is convergent?

   (A) All $x$ except $x = 0$ (B) $|x| = 3$ (C) $-3 \leqq x \leqq 3$ (D) $|x| > 3$
   (E) The series diverges for all $x$.

6. If $\frac{d}{dx}f(x)=g(x)$ and if $h(x)=x^2$, then $\frac{d}{dx}f(h(x))=$

(A) $g(x^2)$ (B) $2x\,g(x)$ (C) $g'(x)$ (D) $2x\,g(x^2)$ (E) $x^2\,g(x^2)$

7. What is $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$?

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\infty$ (E) The limit does not exist.

8. Let $f$ be a function defined for all real $x$. If, for each real number $K$, there exists a $\delta>0$ such that $f(x)>K$ whenever $0<|x|<\delta$, which of the following must be true?

(A) $\lim_{x\to 0}f(x)=\infty$ (B) $\lim_{x\to 0}f(x)=K$ (C) $\lim_{x\to \delta}f(x)=K$

(D) $\lim_{x\to \infty}f(x)=\infty$ (E) $\lim_{x\to \infty}f(x)=K$

9. $\int_0^{+\infty} e^{-2t}dt$

(A) $-\infty$ (B) $-\frac{1}{2}$ (C) $\frac{1}{2}$ (D) 1 (E) $+\infty$

10. What is the area of the closed region bounded by the curve $y=e^{2x}$ and the lines $x=1$ and $y=1$ >?

(A) $\frac{2-e^2}{2}$ (B) $\frac{e^2-3}{2}$ (C) $\frac{3-e^2}{2}$ (D) $\frac{e^2-2}{2}$ (E) $\frac{e^2-1}{2}$

11. Which of the following integrals gives the length of the graph of $y=\sqrt{x}$ between $x=a$ and $x=b$, where $0<a<b$?

(A) $\int_a^b\sqrt{x^2+x}\,dx$ (B) $\int_a^b\sqrt{x+\sqrt{x}}\,dx$ (C) $\int_a^b\sqrt{x+\frac{1}{2\sqrt{x}}}\,dx$

(D) $\int_a^b\sqrt{1+\frac{1}{2\sqrt{x}}}\,dx$ (E) $\int_a^b\sqrt{1+\frac{1}{4x}}\,dx$

12. A point $(x,y)$ is moving along a curve $y=f(x)$. At the instant when the slope of the curve is $-1/3$, the abscissa of the point is increasing at the rate of 5 units per second. The rate of change in units per second of the ordinate of the point is

(A) $-\frac{5}{3}$ (B) $-\frac{1}{3}$ (C) $\frac{1}{3}$ (D) $\frac{3}{5}$ (E) $\frac{5}{3}$

13. What is the nth derivative of $f(x)=\frac{1}{2+x}$ at $x=0$?

(A) $\frac{(-1)^n n!}{2^{n+1}}$ (B) $\frac{(-1)^n n!}{2^n}$ (C) $\frac{(-1)^{n-1}(n-1)!}{2^n}$

(D) $\frac{(-1)^n}{2^n}$ (E) $\frac{(-1)^n}{2^{n+1}}$

14. If $\dfrac{dy}{dx} = y\sec^2 x$ and $y = 5$ when $x = 0$, then $y =$

(A) $e^{\tan x} + 4$ (B) $e^{\tan x} + 5$ (C) $5e^{\tan x}$

(D) $\tan x + 5$ (E) $\tan x + 5e^{x}$

15. Let $a < c < b$ and let $f$ be differentiable on $[a, b]$. Which of the following is NOT necessarily true?

(A) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(B) There exists d in [a, b] such that $f'(d) = \dfrac{f(b) - f(a)}{b - a}$.

(C) $\int_a^b f(x)\,dx \geqq 0$

(D) $\lim_{x \to c} f(x) = f(c)$

(E) If $k$ is a real number, then $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.

16. Let g be a differentiable function on the open interval (0,1), continuous on the closed interval [0,1]. Let g(0) = 1 and g(1) = 0. Then which of the following is NOT necessarily true?

(A) There exists a number $h$ in (0,1) such that g'(h) = -1.

(B) For each number $p$ in [0,2], there exists $h$ in [0,1] such that $g(h) = p$

(C) The Mean Value Theorem applies to g on [0,1].

(D) $\lim_{x \to a} g(x) = g(a)$ for every $a$ in (0, 1).

(E) For every $\epsilon > 0$, , there exists $\delta > 0$ such that if $\left|a - \frac{1}{2}\right| < \delta$, then $\left|g(a) - g\left(\frac{1}{2}\right)\right| < \epsilon$.

17. For all real b, $\int_0^b |2x|\,dx$ is

(A) $-b|b|$ B) $b^2$ (C) $-b^2$ (D) $b|b|$ (E) None of the above

18. The first three nonzero terms in the Taylor series about $x = 0$ of $\ln(1 - x)$ are

(A) $-x - \dfrac{x^2}{2} - \dfrac{x^3}{3}$ (B) $1 - x + \dfrac{x^2}{2}$ (C) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3}$

(D) $-1 + x - \dfrac{x^2}{2}$ (E) $-x + \dfrac{x^2}{2} - \dfrac{x^3}{3}$

19. If F is a continuously differentiable function for all real $x$, then $\lim_{h \to 0} \dfrac{1}{h}\int_a^{a+h} F'(x)\,dx$ is

(A) 0 (B) F(0) (C) F(a) (D) F'(0) (E) F'(a)

20. If $y = x^{(x^2)}$ for $x > 0$, then $\dfrac{dy}{dx} =$

(A) $x^2 \cdot x^{(x^2 - 1)}$ (B) $2x^{(x^2+1)}\ln x$ (C) $x + 2x\ln x$

(D) $x^{(x^2+1)}(1 + 2\ln x)$ (E) $3x^2$

21. $\int_0^1 \text{Arctan } x\, dx$ is

(A) $\frac{\pi}{4}$
(B) $\frac{\pi - 4\ln 2}{4}$
(C) $\frac{\pi + 4\ln 2}{4}$
(D) $\frac{\pi - 2\ln 2}{4}$
(E) $\frac{\pi + 2\ln 2}{4}$

22. $\frac{d}{dx}\int_0^{x^2} \cos^2 t\, dt =$

(A) $\cos^2 x$
(B) $\cos^2 x^2$
(C) $\sin^2 x^2$
(D) $2x\cos^2 x^2$
(E) $x^2 \cos^2 x^2$

23. The area of one loop of the graph of the polar equation $r = 2\sin(3\theta)$ is

(A) $4\int_0^{\frac{\pi}{3}} \sin^2(3\theta)d\theta$
(B) $2\int_0^{\frac{\pi}{3}} \sin(3\theta)d\theta$
(C) $2\int_0^{\frac{\pi}{3}} \sin^2(3\theta)d\theta$
(D) $2\int_0^{\frac{2\pi}{3}} \sin^2(3\theta)d\theta$
(E) $2\int_0^{\frac{2\pi}{3}} \sin(3\theta)d\theta$

24. $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$ is

(A) $\sqrt{3} - 2$
(B) $\frac{\pi}{12}$
(C) $\frac{1}{4}\ln 3$
(D) $\frac{\pi}{6}$
(E) nonexistent

## Section II. Questions Requiring Detailed Answers

Section II consists of six free-response questions. Following are the directions for Section II and the free-response question from the 1990 examination.

*Directions:* Show all your work. Indicate clearly the methods you use because you will be graded on the correctness of your methods as well as on the accuracy of your final answers.

*Notes:* (1) In this examination ln $x$ denotes the natural logarithm of $x$ (that is, logarithm to the base $e$). (2) Unless otherwise specified, the domain of a function $f$ is assumed to be the set of all real numbers $x$ for which $f(x)$ is a real number.

1. A particle starts at time $t = 0$ and moves along the $x$-axis so that its position at any time $t \geq 0$ is given by $x(t) = (t - 1)^3(2t - 3)$.
   (a) Find the velocity of the particle at any time $t \geq 0$
   (b) For what values of t is the velocity of the particle less than zero?
   (c) Find the values of t when the particle is moving and the acceleration is zero.
2. Let $R$ be the region in the $xy$-plane between the graphs of $y = e^x$ and $y = e^{-x}$ from $x = 0$ to $x = 2$.
   (a) Find the volume of the solid generated when R is revolved about <u>$x$-axis.</u>
   (b) Find the volume of the solid generated when R is revolved about the <u>$y$-axis</u>.

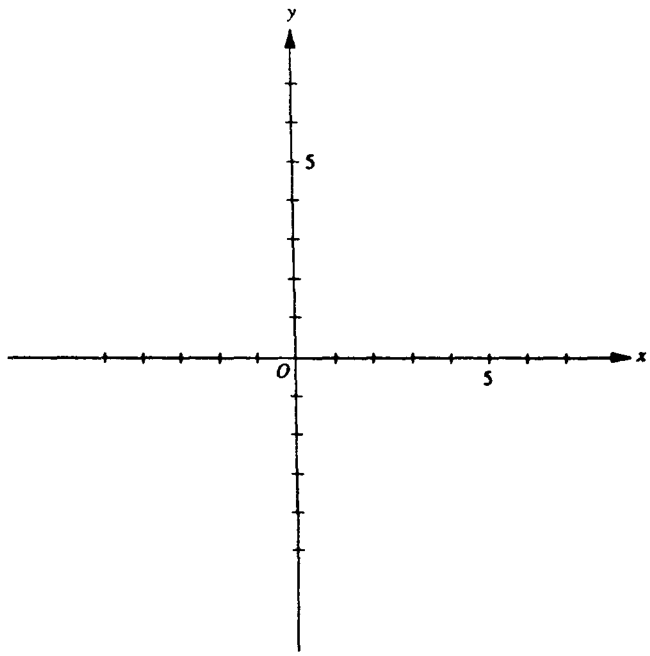
3. Let $f(x) = 12 - x^2$ for $x \geq 0$ and $f(x) \geq 0$.

   (a) The line tangent to the graph of f at point $(k, f(k))$ intercepts the $x$-axis at $x = 4$. What is the value of $k$?

   (b) An isosceles triangle whose base is the interval from (0, 0) to (c, 0) has its vertex on the graph of $f$. For what value of c does the triangle have maximum area? Justify your answer.

4. Let $R$ be the region inside the graph of the polar curve $r = 2$ and outside the graph of the polar curve $r = 2(1 - \sin\theta)$.

   (a) Sketch the two polar curves in the $xy$-plane provided below and shade the region $R$.




   (b) Find the area of $R$.

5. Let $f$ be the function defined by $f(x) = \dfrac{1}{x-1}$.

   (a) Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x = 2$

   (b) Use the result from part (a) to find the first four terms and the general term of the series expansion about $x = 2$ for $\ln|x - 1|$.

   (c) Use the series in part (b) to compute a number that differs from $\ln\frac{3}{2}$ by less than 0.05. Justify your answer.

6. Let $f$ and $g$ be continuous functions with the following properties.

   (i) $g(x) = A - f(x)$, where A is a constant

   (ii) $f(x)dx = \quad g(x)dx$

   (iii) $\int_2^3 f(x)dx = -3A$

   (d) Find $\int_1^3 f(x)dx$ in term of A

   (e) Find the average value of $g(x)$ in terms of A, over the interval [1, 3].

   (f) Find the value of $k$ if $\int_0^1 f(x+1)dx = kA$.

*Answer key for multiple-choice section:*

**Calculus BC**

1–B, 2–B, 3–E, 4–E, 5–D, 6–D, 7–A, 8–A, 9–C, 10–B, 11–E, 12–A, 13–A
14–C, 15–C, 16–B, 17–D, 18–A, 19–E, 20–D, 21–D, 22–D, 23–C, 24–B