NOTE 1. LINEAR SYSTEM

Finite vs. Infinite Duration Signals

A discrete signal $x[n]$ is finite duration if there exists two integers $-\infty < N_1 \le N_2 < \infty$, such that $x[n] \neq 0$ only for $N_1 \le n \le N_2$.

Otherwise, it is of infinite duration.

Right-sided, Left-sided, and Two-sided Signals

The terms apply only to infinite duration signals.

If there exists an integer N_1 such that $x[n] \neq 0$ only for $n \ge N_1$, then x[n] is right-sided.

If there exists an integer N_2 such that $x[n] \neq 0$ only for $n \le N_2$, then x[n] is left-sided.

If x[n] is neither right-sided nor left-sided, then it is two-sided.

Causal and Anti-Causal Signals

A signal x[n] is causal if $x[n] = 0$ for all $n < 0$.

A signal x[n] is anti-causal if $x[n] = 0$ for all $n > 0$.

4. Zero phase, linear phase and nonlinear phase signals

Zero Phase: Left-right symmetry around sample number zero.

Linear Phase: Left-right symmetry around sample number other than zero. Linear phase signal can be changed to zero phase one by simple shifting.

Nonlinear Phase: without left-right symmetry.

5. Time Delay

System (Transform, Mapping): A Relationship between Input and Output.

7. Linear System S can be characterized by the following conditions:

Homogeneity:

For constant k,

if system S satisfies $S(x[n]) = y[n]$,

then $S(kx[n]) = kS(x[n]) = ky[n]$.

Dr. Zhuoer Shi, zshi@uh.edu, Department of Physics, University of Houston

Additivity:

If $S(x_1[n]) = y_1[n]$, and $S(x_2[n]) = y_2[n]$

Then $S(x_1[n] + x_2[n]) = S(x_1[n]) + S(x_2[n]) = y_1[n] + y_2[n]$

Commutative:

for linear systems A and B

$A(B(x[n])) = B(A(x[n]))$

Any System violates above-mentioned three rules, is a nonlinear system.

8. Shift (Time) Invariance System:

if $S(x[n]) = y[n]$, then $S(x[n + i]) = y[n + i]$.

9. Energy of signal x[n]: $E = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$.

NOTE 2. Convolution

Convolution – most important technique in DSP, by combining two (or more) signals to form a third output.

Signal Filtering (Estimation), Function Approximation
Interpolation
Prediction (extrapolation)

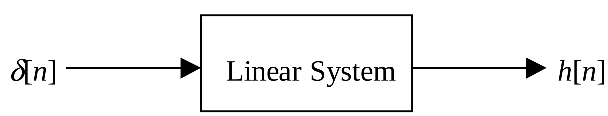
Definition 1: Delta Function (Unit Impulse) $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & others \end{cases}$

Standard Function $u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & others \end{cases}$

$\delta[n] = u[n] - u[n - 1]$

$$u[n] = \sum_{k=0}^{\infty} \delta(n-k)$$

Definition 2: Impulse response $h[n]$ of linear system S. The signal that exits a system when a Delta Function (unit impulse) is the input.



$h[n] = S(\delta[n])$

Any sequence $x[n]$ can be represented by weighted Delta function

$$x[n] = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

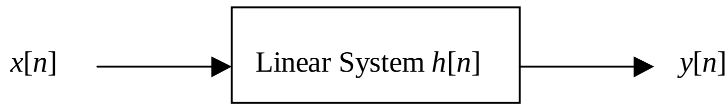
Definition 3: Periodic Sequence

$x[n] = x[n + N]$

Example:

$sin[2\pi f(n+N)] = sin[2\pi fn]$, f is normalized frequency.

Definition 4: Convolution $y[n] = h[n] * x[n]$



The impulse response of the system expresses the relation between input and output.

$$y[i] = \sum_{j=-\infty}^{\infty} h[j]x[i-j]$$

Convolution Sum (Linear System).

NOTE 3. Convolution Properties

Properties of Delta function: $\delta[n]$ is identity for convolution,

$x[n] * \delta[n] = x[n]$

$x[n] * k\delta[n] = kx[n]$

$x[n] * \delta[n+i] = x[n+i]$

Properties of Convolution (Convolution is a linear system)

Commutative: $a[n] * b[n] = b[n] * a[n]$

Associative: $\{a[n] * b[n]\} * c[n] = a[n] * \{b[n] * c[n]\}$

Distributive: $a[n] * b[n] + a[n] * c[n] = a[n] * \{b[n] + c[n]\}$

Transference between the Input and Output

Suppose $y[n] = x[n] * h[n]$,

If L is a linear system and $x_1[n] = L\{x[n]\}$, $y_1[n] = L\{y[n]\}$

Then $y_1[n] = x_1[n] * h[n]$

5. (Auto-Regression): $y[n] = \sum_{k=0}^{M-1} a[k]x[n-k]$

For Example: $y[n] = x[n] - x[n-1]$ (first difference)

6. (Moving Average): $y[n] = b[0]x[n] + \sum_{k=1}^{M-1} b[k]y[n-k]$

For Example: $y[n] = x[n] + y[n-1]$ (running sum)

7. Central Limit Theorem: $y[n] = x[n] * x[n] * \ldots * x[n]$

8. Correlation: Convolution = Cross-Correlation $c[n] = a[n] * b[-n]$

Autocorrelation: $c[n] = a[n] * a[-n]$

Detector (Not Restoration).

Low-Pass Filter $h[n]$: $\sum_n h[n] \neq 0$, $\sum_n (-1)^n h[n] = 0$

Cutoff the high-frequency components (undulation, pitches), smooth the signal

High-pass Filter $h[n]$: $\sum_{n} h[n] = 0$

Only preserve the quick undulation terms

Delta function: All-pass filter $x[n] = x[n] * \delta[n]$

NOTE 4. Fourier Representation

Any continuous periodic signal ⇔ sum of properly chosen sinusoidal functions

Gibbs Effects: Problem of non-smooth continuous

4 Categories of Fourier Representation ($\omega = 2\pi f$)

(1) Aperiodic Continuous Fourier Transform

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

Aperiodic

Frequency $f \in R$ Continuous

Time (Shift) $t \in R$ Continuous

(2) Periodic Continuous Fourier Series (Frequency Response)

$$X(f) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j2\pi fn}$$

Use to Characterize the Filters and Signals. Response

Periodic $X(f) = X(f+1)$

Frequency $f \in$ R Continuous

Time (Shift) $k \in$ Z Discrete

Aperiodic Discrete Time Fourier Transform

$$X(k) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi k\Delta ft} dt$$

Aperiodic

Frequency $k \in$ Z Discrete

Time (Shift) $t \in$ R Continuous

(4) Periodic Discrete Fourier Transform

$$X[k] = \sum_{i=0}^{N-1} x[i] e^{-j2\pi ki/N}$$

$$e^{-j2\pi ki/N} = \cos(2\pi ki/N) - j\sin(2\pi ki/N)$$

$$X[k] = \sum_{i=0}^{N-1} x[i]\cos(2\pi ki / N) - j\sum_{i=0}^{N-1} x[i]\sin(2\pi ki / N)$$

Periodic	$X(k + N) = X(k)$	
Frequency	$k \in Z$	Discrete
Time (Shift)	$n \in Z$	Discrete

Frequency Response Frequency Transform of Sequence $h[n]$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-jk\omega}$$

Inverse Frequency Transform

$$h[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{jk\omega}d\omega$$

Continuous

Periodic $H(\omega + 2\pi) = H(\omega)$

≠ Discrete Fourier Transform (DFT)

Ideal Low-pass Filter

$$H(\omega) = \begin{cases} 1, & |\omega| \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| \le \frac{\pi}{2} \end{cases}$$

Inverse Frequency Response

$$h[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{jk\omega}d\omega = \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} e^{jk\omega}d\omega = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}$$

Not Causal

Technically CAN NOT be Realized

Cut-off cause Gibbs Oscillation

Frequency Transform of Convolution

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$Y(\omega) = X(\omega)H(\omega)$$

NOTE 5. Properties of Frequency Response

Periodic Continuous Fourier Series

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

Frequency Response of Convolution

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

$$\Downarrow$$

$Y(\omega) = X(\omega)H(\omega)$

$H(\omega) = Y(\omega) / X(\omega)$ *Transfer Function* of the Linear System

$|H(\omega)| = [\mathrm{Im}\, H(\omega)^2 + \mathrm{Re}\, H(\omega)^2]^{1/2}$ *Magnitude* of Transfer Function

$\mathrm{Arg}(H(\omega)) = \arctan(\mathrm{Im}\, H(\omega)/\mathrm{Re}\, H(\omega))$ *Phase* of Transfer Function

Even Symmetry around Point Zero $x[n] = x[-n]$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

$$= x[0] + \sum_{n=1}^{\infty} (x[n] e^{-j\omega n} + x[-n] e^{j\omega n})$$

$$= x[0] + \sum_{n=1}^{\infty} x[n](e^{-j\omega n} + e^{j\omega n})$$

$$= x[0] + 2\sum_{n=1}^{\infty} x[n] \cos(\omega n)$$

Real Spectrum

Phase $\mathrm{Arg}(H(\omega)) = 0$ *Zero-phase Signal*

Odd Symmetry around Point Zero $x[n] = -x[-n]$, $x[0] = 0$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

$$= x[0] + \sum_{n=1}^{\infty}(x[n]e^{-j\omega n} + x[-n]e^{j\omega n})$$

$$= \sum_{n=1}^{\infty} x[n](e^{-j\omega n} - e^{j\omega n})$$

$$= -2j\sum_{n=1}^{\infty} x[n]\sin(\omega n)$$

Complex Spectrum

Phase Arg($H(\omega)$) = $-\pi/2$

Even Symmetry around sample number N other than zero $x[N+n] = x[N-n]$

$$X(\omega) = \sum_{k=-\infty}^{+\infty} x(k)e^{-j\omega k}$$

$$= x[N]e^{-j\omega N} + \sum_{k=-\infty}^{N-1} x[k]e^{-j\omega k} + \sum_{k=N+1}^{\infty} x[k]e^{-j\omega k}$$

$$= x[N]e^{-j\omega N} + e^{-j\omega N}\sum_{n=1}^{\infty} x[n+N](e^{j\omega n} + e^{-j\omega n})$$

$$= e^{-j\omega N}\{x[N] + 2\sum_{n=1}^{\infty} x[n+N]\cos(\omega n)\}$$

Complex Spectrum

Phase Arg($H(\omega)$) = $-N\omega$ Linear-phase Signal

Odd-symmetric signal Generalized linear-pase signal

Even/Odd Decomposition of any Real Signal $x[n]$

Even Symmetry $x_e[n] = (x[n] + x[-n]) / 2$

$x_e[n] = x_e[-n]$

Odd Symmetry $x_o[n] = (x[n] - x[-n])/2$

$x_o[n] = -x_o[-n]$

Even Symmetry Sequence $x_e[n]$ generate real spectrum

Odd Symmetry Sequence $x_o[n]$ generate complex spectrum

(4) $x[n] \longrightarrow X(\omega)$

$\searrow$

$x[-n] \longrightarrow \overline{X(\omega)}$

NOTE 6. Digital Fourier Transform (DFT)

Periodic Continuous Fourier Series of $x[n]$, $n = 0, N-1$

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

Let $\omega = 2\pi k / N, \quad k = 0, N-1$

$$\boxed{X[k] = X(\omega)|_{\omega = 2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}}$$

Periodic Discrete Fourier Transform of Real Signal

$$e^{-j2\pi ki/N} = \cos(2\pi ki/N) - j\sin(2\pi ki/N)$$

$$\boxed{\operatorname{Re} X[k] = \sum_{i=0}^{N-1} x[i]\cos(2\pi ki/N)} \qquad \boxed{\operatorname{Im} X[k] = -\sum_{i=0}^{N-1} x[i]\sin(2\pi ki/N)}$$

Periodic Character

$$\boxed{X[k+N] = X[k]}$$

Inverse Discrete Fourier Transform

$$\sum_{k=0}^{N-1} X[k] e^{j2\pi km/N} = \sum_{k=0}^{N-1}\{\sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}\} e^{j2\pi km/N} = \sum_{n=0}^{N-1} x[n]\{\sum_{k=0}^{N-1} e^{j2\pi k(m-n)/N}\}$$

Because $$\sum_{k=0}^{N-1} e^{j2\pi k(m-n)/N} = \begin{cases} N, & m = n \\ 0, & m \neq n \end{cases}$$

Then $$\sum_{k=0}^{N-1} X[k] e^{j2\pi km/N} = Nx[m]$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N}$$

Simplified Algorithm

$$X[N-k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi(N-k)n/N} = \sum_{n=0}^{N-1} x[n] e^{j2\pi kn/N}$$

Re $X[N - k]$ = Re $X[k]$ Im $X[N-k]$ = -Im $X[k]$

$|X[N - k]| = |X[k]|$, $\mathrm{Arg}(X[N - k]) = -\mathrm{Arg}X[k]$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{Re}\, X[k] \cos(2\pi kn/N) - \frac{1}{N}\sum_{k=0}^{N-1} \mathrm{Im}\, X[k] \sin(2\pi kn/N)$$

$\cos(2\pi(N - k)n / N) = \cos(2\pi kn / N)$

$\sin(2\pi(N - k)n / N) = -\sin(2\pi kn / N)$

$x[n] = \{\mathrm{Re}\, X[0] + \mathrm{Re}\, X[N / 2] \cos(n\pi)\} / N$

$$+\frac{2}{N}\sum_{k=1}^{N/2-1} \mathrm{Re}\, X[k] \cos(2\pi kn/N) - \frac{2}{N}\sum_{k=1}^{N/2-1} \mathrm{Im}\, X[k] \sin(2\pi kn/N)$$

NOTE 7: CHARACTERISTICS OF DFT

Linear: If $x[n] = ax_1[n] + bx_2[n]$, then $X[k] = aX_1[k] + bX_2[k]$.

Circular Shift:

Time: $DFT(x[n-m]) = e^{-j2\pi km/N} X[k]$

Frequency: $IDFT(X[k-l]) = e^{j2\pi nl/N} X[k]$

Circular Convolution

If $y[n] = x_1[n] * x_2[n] = \sum_{m=0}^{N-1} x_1[m]x_2[n-m]$, then $Y[k] = X_1[k]X_2[k]$.

If $Y[k] = X_1[k] * X_2[k] = \frac{1}{N}\sum_{m=0}^{N-1} X_1[m]X_2[k-m]$, then $y[n] = x_1[n]x_2[n]$.

Non-Overlap Condition

$x_1[n]$: L non-zero,

$x_2[n]$: M non-zero

The zero-extended period duration $N \geq L + M - 1$

Inverse DFT (can be implemented using the same program of DFT)

$$x[n] = \frac{1}{N}\sum_{m=0}^{N-1} X[m]e^{j2\pi mn/N} = \frac{1}{N}\left(\sum_{m=0}^{N-1} X^*[m]e^{-2\pi mn/N}\right)^*$$

DFT: $X[k] = \sum_{m=0}^{N-1} x[m]e^{-j2\pi mn/N}$

Symmetry Theorem

If $DFT(x[n]) = X[k]$, then $DFT(x[-n]) = X[-k]$

$$DFT\left(\frac{X[k]}{N}\right) = x[-n]$$

Initial State

Frequency Domain $X[0] = \sum_{n=0}^{N-1} x[n]$

Time Domain $x[0] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]$

Zero Extension

Time Domain

If $g[n] = \begin{cases} x[n], & 0 \leq n \leq N-1 \\ 0, & N \leq n \leq rN-1 \end{cases}$, then $G[k] = DFT(g[n]) = X\left(\frac{k}{r}\right)$

Frequency Domain

If $Y[k] = \begin{cases} X[k], & 0 \leq k \leq N-1 \\ 0, & N \leq n \leq rN-1 \end{cases}$, then $y[n] = IDFT(Y[k]) = \frac{1}{r}x\left(\frac{n}{r}\right)$, $k = 0, rN-1$.

Discrete Cosine Transform
Orthogonal Transform
Symmetry-Periodic Extension
$x[-n] = x[n]$

JPEG (Joint Photographic Expert Group)
MPEG (Moving Picture Expert Group)

DCT $$X[k] = \sqrt{\frac{2}{N}} a[k] \sum_{n=0}^{N-1} x[n] \cos\left(\frac{(2n+1)k\pi}{2N} \right)$$

IDCT $$x[n] = \sqrt{\frac{2}{N}} \sum_{k=0}^{N-1} a[k] X[k] \cos\left(\frac{(2n+1)k\pi}{2N} \right)$$

where $$a[k] = \begin{cases} \frac{\sqrt{2}}{2}, & k = 0 \\ 1, & else \end{cases}$$

DCT and IDCT share the same transformation kernel
Real spectrum, can be implemented by 2N-dot FFT, smaller boundary effects.

NOTE 8 ADDITIONAL STATEMENT ON DFT

Parseval Relation

$$\sum_{n=0}^{N-1}|x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$

$$= \frac{1}{N}\left(|X[0]|^2 + \left|X\left[\frac{N}{2}\right]\right|^2 + 2\sum_{k=1}^{N/2-1}|X[k]|^2\right)$$

Let $X[0] = \frac{1}{\sqrt{2}}X[0]$, $X[N/2] = \frac{1}{\sqrt{2}}X[N/2]$

$$\sum_{n=0}^{N-1}|x[n]|^2 = \frac{2}{N}\sum_{k=0}^{N/2}|X[k]|^2$$

New IDFT Equation

$$x[n] = \frac{1}{N}\left\{\mathrm{Re}\,X[0] + \mathrm{Re}\,X[N/2](-1)^n + 2\sum_{k=1}^{N/2-1}\mathrm{Re}\,X[k]\cos(2\pi kn/N) - 2\sum_{k=1}^{N/2-1}\mathrm{Im}\,X[k]\sin(2\pi kn/N)\right\}$$

Let $X[0] = \frac{1}{\sqrt{2}}X[0]$, $X[N/2] = \frac{1}{\sqrt{2}}X[N/2]$

$$x[n] = \frac{2}{N}\sum_{k=0}^{N/2}\{\mathrm{Re}\,X[k]\cos(2\pi kn/N) - \mathrm{Im}\,X[k]\sin(2\pi kn/N)\}$$

Convolution Equivalence of DFT

Let $h_k[n] = e^{j2\pi kn/N}$

$$X[k] = \sum_{i=0}^{N-1}x[i]h_k[0-i] = x[n] * h_k[n]|_{n=0}$$

Time Sampler of All Filters is 0.

Frequency Response

$$H_k[\omega] = \sum_{n=0}^{N-1}h_k[n]e^{-jn\omega} = \sum_{n=0}^{N-1}e^{-jn\omega}e^{j2\pi nk/N} = \frac{1-e^{-jN\omega}e^{j2\pi k}}{1-e^{-j\omega}e^{j2\pi k/N}}$$

Magnitude Response

$$|H_k[\omega]| = \frac{\sin(N\omega/2 - \pi k)}{\sin(\omega/2 - \pi k/N)}$$

Center Frequency $\omega_k = \frac{2\pi k}{N}$.

Frequency Leakage: when signal frequency ω is between ω_k and ω_{k+1}.

NOTE 9 FOURIER TRANSFORM PAIR

1. Frequency Response (FR)

- Delta Function

If $x[n] = \delta[n]$, then $X(\omega) = 1$.
If $x[n] = \delta[n - m]$, then $X(\omega) = e^{-j\omega m}$.

- Square Function

Example 1:

If $x[n] = 1$, $n = 0, M$-1, then $X(\omega) = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}}$ and $|X(\omega)| = \frac{\sin(\omega M/2)}{\sin(\omega/2)}$

Example 2:

If $x[n] = 1$, $|n| = 0, M$, then $X(\omega) = \frac{\sin(\omega(2M+1)/2)}{\sin(\omega/2)}$.

If $y[n] = x[n - M]$, then $Y(\omega) = X(\omega)\, e^{-j\omega M}$
If $y[n] = x[n]\, e^{jan}$, then $Y(\omega) = X(\omega - a)$

2. Discrete Fourier Transform (DFT)

- Delta Function

Time Region

If $x[n] = \delta[n]$, n=0, $N - 1$, then $X[k]$ = DFT($\delta[n]$) = 1, k=0, $N - 1$.

If $x[n] = \delta[n - m]$, n=0, $N - 1$, then $X[k]$ = DFT($\delta[n - m]$) = $e^{-j2\pi km/N}$, k=0, $N - 1$.
Magnitude $|X[k]| = 1$,
Phase Arg($X[k]$) = arctan[tan($-2\pi km/N$)]

Frequency Region

If $X[k] = \delta[k]$, k = 0, $N - 1$, then $x[n] = IDFT(\delta[k]) = \frac{1}{N}$, n = 0, $N - 1$.

If $X[k] = \delta[k - M]$, k = 0, $N - 1$, then $x[n] = IDFT(\delta[k]) = \frac{1}{N} e^{j2\pi nM/N}$, n = 0, $N - 1$.

- Square Function

Time Region:

Example 1: $x[n] = 1$, $n = 0, M - 1$

$$X[k] = \sum_{n=0}^{M-1} e^{-j2\pi nk/N} = \frac{1-e^{-j2\pi kM/N}}{1-e^{-j2\pi k/N}} = e^{j\pi k(1-M)/N} \frac{\sin(\pi kM/N)}{\sin(\pi k/N)}$$

$$|X[k]| = \frac{\sin(\pi kM/N)}{\sin(\pi k/N)}$$

Example 2:

If $x[n] = 1$, $n = 0, M$ and $x[n] = 1$, $n = N - M, N - 1$,

then $X[k] = \dfrac{\sin(\pi k(2M+1)/N)}{\sin(\pi k/N)}$

Frequency Region:

If $X[k] = 1$, when $k = 0, M$; and $X[k] = 1$, when $k = N - M, N - 1$,

then $x[n] = \dfrac{1}{N}\left(\sum_{k=0}^{M} e^{j2\pi kn/N} + \sum_{k=N-M}^{N-1} e^{2\pi kn/N}\right) = \dfrac{1}{N}\dfrac{\sin(\pi n(2M+1)/N)}{\sin(\pi n/N)}$

$\mathrm{DFT}(x[n - M]) = X[k]e^{-j2\pi kM/N}$

$\mathrm{IDFT}(X[k - M]) = x[n]e^{j2\pi nM/N}$

NOTE 10. FAST FOURIER TRANSFORM

FFT: Fast implementation of DFT

Signal Length $N = 2^L$

Let $x_1[r] = x[2r]$, $x_2[r] = x[2r + 1]$, $r = 0, N/2 - 1$

$$
\begin{aligned}
& X[k] \\
&= DFT(x[n]) \\
&= \sum_{r=0}^{N/2-1} x[2r] e^{-j2\pi(2r)k/N} + \sum_{r=0}^{N/2-1} x[2r+1] e^{-j2\pi(2r+1)k/N} \\
&= \sum_{r=0}^{N/2-1} x[2r] e^{-j2\pi rk/(N/2)} + e^{-j2\pi k/N} \sum_{r=0}^{N/2-1} x[2r+1] e^{-j2\pi rk/(N/2)} \\
&= X_1[k] + X_2[k] e^{-j2\pi k/N}
\end{aligned}
$$

($N/2$ dot FFT !!!)

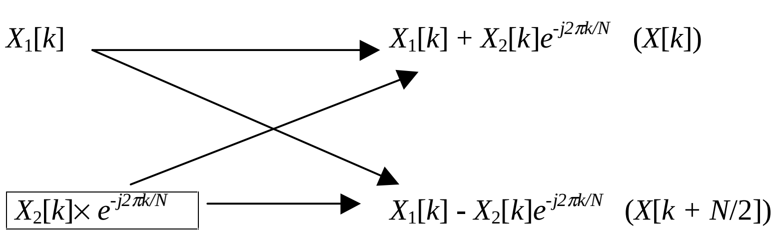
1. For $k = 0, N/2 - 1$,

$$\boxed{X[k] = X_1[k] + X_2[k] e^{-j2\pi k/N}, \quad k = 0, N/2 - 1}$$

2. For $k = N/2, N - 1$,
   Because
   $X_1[k + N/2] = X_1[k]$, $X_2[k+N/2]=X_2[k]$, $e^{-j2\pi(k + N/2)/N} = -e^{-j2\pi k/N}$

$$\boxed{X[k + N/2] = X_1[k] - X_2[k] e^{-j2\pi k/N}, \quad k = 0, N/2 - 1}$$

Butterfly Algorithm



One Butterfly: generates 2 Dots of DFT
Each Layer: $N/2$ Butterflies

Each Butterfly: 1 multiply + 2 add
Layer Number: $L = \log_2 N$

Calculation Efficiency:

1. FFT Cost = $(N/2) \times L \times$ (1 Multiply + 2 Add) = $\frac{N}{2}\log_2 N$ Multiply + $N\log_2 N$ Add

2. DFT Cost = N^2 Multiply + $N(N - 1)$ Add

NOTE 11. FFT Application

Fast Convolution

For $x[n] * h[n]$, $x[n]$, $n = 0, M - 1$ and $h[n]$, $n = 0, L - 1$.

1. To Avoid Overlap, Zero-Extention

Choose $N = 2^J \geq (M + L - 1)$

$$x[n] = \begin{cases} x[n], & n = 0, M-1 \\ 0, & n = M, N-1 \end{cases}$$

$$h[n] = \begin{cases} h[n], & n = 0, L-1 \\ 0, & n = L, N-1 \end{cases}$$

2. Do FFT

$X[k] = \text{FFT}(x[n])$, $H[k] = \text{FFT}(h[n])$, $k = 0, N - 1$

3. Multiplication $Y[k] = X[k]H[k]$

4. IFFT $y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{j2\pi kn/N}$

$$= \overline{\sum_{k=0}^{N-1}(\overline{Y[k]}/N)e^{-j2\pi kn/N}}$$

$$y[n] = \overline{FFT(\overline{Y[k]}/N)}$$

Calculation Efficiency Comparison

Traditional Convolution $y[n] = \sum_{i=0, L-1} x[n-i]h[i]$, $n = 0, L + M - 2$.

Multiply: $(L + M - 1) \times L$

Add: $(L + M - 1) \times (L - 1)$

FFT Convolution

(1) $\text{FFT}(x[n]) + \text{FFT}(h[n]) + \text{FFT}(\overline{Y[k]/N})$

Multiply: $(3N / 2) \log_2 N$

Add: $3N\log_2 N$

(2) $(X[k]H[k]) + (\overline{Y[k] / N})$

Multiply: $2N$

Fast Correlation

$x[n]$, $y[n]$, $n = 0, M - 1$

$z[n] = x[n] * y[-n] = \sum_{i = 0, M - 1} x[i]y[i + n]$

Same Routine! (Zero-Extension + FFT)

Just $Z[k] = X[k]\overline{Y[k]}$

Calculation Efficiency

Traditional Correlation (Only Count Multiply!!)
Cost = $M(2M - 1)$ Multiply

FFT Correlation
Cost= $(3N / 2) \log_2 N + 2N$ Multiply

Worst Case: Signal length of $x[n]$, $y[n]$
$M = 2^L + 1$

Then Extended Length: $N \geq 2M - 1 = 1 + 2^{L+1}$
Thus, $N = 2^{L+2}$

FFT Cost = $(6L + 20)\, 2^L$ Multiply,
while
Traditional Cost = $(1 + 2^L)(1 + 2^{L+1})$ Multiply

When $L \geq 5$, $M \geq 33$ dots, ($N \geq 128$)
FFT Method will be more Efficient than Traditional calculation.

NOTE 12. MULTIRESOLUTION SPACES

Operator:

Down Sampling: If $y[n] = x[Mn]$, then $Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X(\frac{\omega - 2\pi k}{M})$

Up Sampling: $y[l] = \begin{cases} x[n], & l = Mn \\ 0, & others \end{cases}$, then $Y(\omega) = X(M\omega)$

Scaling Function $\phi(x)$ is an Approximating Function .

If $\{\phi(x-n) \mid n \in Z\}$ is a Rieze basis of V_0. Namely, exist unique sequence $\{C_0(n) \mid n \in Z\}$, for any function $f(x) \in V_0$, make $f(x) = \sum_n C_0[n]\phi(x-n)$.

Two-Scale Difference Equation $\phi(x/2) = 2\sum_n h[n]\phi(x-n)$

Construct the Multiresolution Approximation

(1) $V_m \subset V_{m-1}$

(2) $f(x) \in V_m \Leftrightarrow f(2x) \in V_{m-1}$

(3) $f(x) \in V_m \Rightarrow f(x - 2^m n) \in V_m$

(4) Exist two constants A and B, $0 < A \le B$, that
$A \,\| f \| \le \sum_n |C_0[n]|^2 \le B \,\| f \|$

Multi-Resolution Approximation (From Fine to Coarse)

For any $f(x) \in L^2(R)$,

$$P_0 f(x) = \sum_n C_0[n]\phi(x-n), \quad for\ V_0$$

$$P_m f(x) = \sum_n C_m[n]\phi(2^{-m}x - n), \quad for\ V_m$$

Ideal Low-Pass Function is a Scaling function

$$\phi[x] = Sinc(\pi x)$$

$$\phi(x/2) = \sum_n Sinc\left(\frac{\pi n}{2}\right)\phi(x-n)$$

$$h[n] = \frac{1}{2} Sinc\left(\frac{\pi n}{2}\right)$$

NOTE 13. WAVELET TRANSFORM

$$\phi_{m,n}(x) = 2^{-m/2}\phi(2^{-m}x - n), \qquad \psi_{m,n}(x) = 2^{-m/2}\psi(2^{-m}x - n).$$
$$\phi(x/2) = 2\sum_n h[n]\phi(x-n), \qquad \psi(x/2) = 2\sum_n g[n]\phi(x-n).$$

$$\sum_n h[n] = 1, \qquad \sum_n g[n] = 0$$

Scaling Transform

$$C_m[n] = \left\langle f(x), \phi_{m,n}(x) \right\rangle = \int_{-\infty}^{\infty} f(x)\phi_{m,n}(x)dx$$

Let $y = 2^{-j}x - k$

$$\phi_{m,n}(x) = 2^{-m/2}\phi(2^{-m}x - n)$$
$$= 2^{-m/2}\phi(y) = 2^{-m/2}\left\{2\sum_k h[k]\phi(2y-k)\right\}$$
$$= \sqrt{2}\sum_k h[k]\left\{2^{-(m-1)/2}\phi(2^{-(j-1)}x - 2n - k)\right\}$$
$$= \sqrt{2}\sum_k h[k]\phi_{m-1,2n+k}(x)$$

$$C_m[n] = \left\langle f(x), \phi_{m,n}(x) \right\rangle$$
$$= \sqrt{2}\sum_k h[k]\left\langle f(x), \phi_{m-1,2n+k}(x) \right\rangle$$
$$= \sqrt{2}\sum_k h[k]C_{m-1}[2n+k]$$

Wavelet Transform

$$D_m[n] = \left\langle f(x), \psi_{m,n}(x) \right\rangle = \int_{-\infty}^{\infty} f(x)\psi_{m,n}(x)dx$$

$D_j[k] = \langle f, \psi_{j,k} \rangle$

$$\psi_{m,n}(x) = 2^{-m/2}\psi(2^{-m}x - n)$$
$$= 2^{-m/2}\psi(y) = 2^{-m/2}\left\{2\sum_k g[k]\phi(2y-k)\right\}$$
$$= \sqrt{2}\sum_k g[k]\left\{2^{-(m-1)/2}\phi(2^{-(j-1)}x - 2n - k)\right\}$$
$$= \sqrt{2}\sum_k g[k]\phi_{m-1,2n+k}(x)$$

$$D_m[n] = \left\langle f(x), \psi_{m,n}(x)\right\rangle$$
$$= \sqrt{2}\sum_k g[k]\left\langle f(x), \phi_{m-1,2n+k}(x)\right\rangle$$
$$= \sqrt{2}\sum_k g[k]C_{m-1}[2n+k]$$

Properties of Orthonormal Bases

$$\left\langle \phi(x), \phi(x-k)\right\rangle = \delta[k]$$
$$\left\langle \phi(x), \psi(x-k)\right\rangle = 0$$
$$\left\langle \psi(x), \psi(x-k)\right\rangle = \delta[k]$$

$$\int_{-\infty}^{\infty}\phi(x)dx = 1, \qquad \int_{-\infty}^{\infty}\psi(x)dx = 0$$

If $f(x) \in V_0$,

$$f(x) = \sum_i C_0[i]\phi_{0,i}(x) = \sum_n C_1[n]\phi_{1,n}(x) + \sum_l D_1[l]\psi_{1,l}(x)$$

NOTE 14. DIGITAL IMPLEMENTATION

DISCRETE WAVELET DECOMPOSITION

$$C_m[n] = \left\langle f(x),\, \phi_{m,n}(x) \right\rangle = \left\langle f(x),\, \sqrt{2}\sum_i h[i]\phi_{m-1,\,i+2n}(x) \right\rangle$$
$$= \sqrt{2}\sum_i C_{m-1}[i]h[i-2n]$$
$$D_m[n] = \left\langle f(x),\, \psi_{m,n}(x) \right\rangle = \left\langle f(x),\, \sqrt{2}\sum_i g[i]\phi_{m-1,\,i+2n}(x) \right\rangle$$
$$= \sqrt{2}\sum_i C_{m-1}[i]g[i-2n]$$

WAVELET RECONSTRUCTION

$$V_m \oplus W_m = V_{m-1}$$
$$P_m f \Rightarrow V_m \text{ , } Q_m f \Rightarrow W_m \;\; P_m f + Q_m f = P_{m-1} f$$

$$P_{m-1} f = \sum_n C_{m-1}[n]\phi_{m-1,n}(x)$$
$$= P_m f + Q_m f$$
$$= \sum_i C_m[i]\phi_{m,i}(x) + \sum_l D_m[l]\phi_{m,l}(x)$$

$$C_{m-1}[n] = \left\langle \sum_i C_{m-1}[i]\phi_{m-1,i}(x),\;\; \phi_{m-1,n}(x) \right\rangle$$
$$= \sum_i C_m[i]\left\langle \phi_{m,i}(x), \phi_{m-1,n}(x) \right\rangle + \sum_l D_m[l]\left\langle \psi_{m,l}(x), \phi_{m-1,n}(x) \right\rangle$$
$$= \sum_i C_m[i]\left(\sqrt{2}\sum_k h[k]\left\langle \phi_{m-1,k+2i}(x), \phi_{m-1,n}(x) \right\rangle \right) + \sum_l D_m[l]\left(\sqrt{2}\sum_k g[k]\left\langle \psi_{m-1,k+2l}(x), \phi_{m-1,n}(x) \right\rangle \right)$$
$$= \sqrt{2}\left(\sum_i C_m[i]h[n-2i] + \sum_l D_m[l]g[n-2l] \right)$$

Decomposition $C_m[n] = \sqrt{2}\sum_i C_{m-1}[i]h[i-2n]$, $D_m[n] = \sqrt{2}\sum_i C_{m-1}[i]g[i-2n]$.

Reconstruction $C_{m-1}[n] = \sqrt{2}\left(\sum_i C_m[i]h[n-2i] + \sum_l D_m[l]g[n-2l] \right)$.

For Computer Implementation

Decomposition $C_m[n] = \sum_i C_{m-1}[i]h[i-2n]$, $D_m[n] = \sum_i C_{m-1}[i]g[i-2n]$.

Reconstruction $C_{m-1}[n] = 2\times\left(\sum_i C_m[i]h[n-2i] + \sum_l D_m[l]g[n-2l]\right)$

Perfect Reconstruction Condition

$$A(\omega) = H(\omega)\left[H^*(\omega)X(\omega) + H^*(\omega-\pi)X(\omega-\pi)\right]$$
$$B(\omega) = G(\omega)\left[G^*(\omega)X(\omega) + G^*(\omega-\pi)X(\omega-\pi)\right]$$

For Perfect Reconstruction:

$$X(\omega) = A(\omega) + B(\omega)$$
$$= X(\omega)\left[H(\omega)H^*(\omega) + G(\omega)G^*(\omega)\right] + X(\omega+\pi)\left[H(\omega)H^*(\omega-\pi) + G(\omega)G^*(\omega-\pi)\right]$$

$$|H(\omega)|^2 + |G(\omega)|^2 = 1$$
$$H(\omega)H^*(\omega-\pi) + G(\omega)G^*(\omega-\pi) = 0$$

Conjugate Quadrature Filter (CQF)

Let

$$G(\omega) = H^*(\omega-\pi)\left[\pm e^{-j\omega(2M+1)}\right]$$
$$G(\omega-\pi) = -H^*(\omega)\left[\pm e^{j\omega(2M+1)}\right]$$

Then $H(\omega)H^*(\omega-\pi) + G(\omega)G^*(\omega-\pi) = 0$

$$|H(\omega)|^2 + |H(\omega-\pi)|^2 = 1$$
$$|G(\omega)|^2 + |G(\omega-\pi)|^2 = 1$$

CQF Example (Let $M = 0$)

$$G(\omega) = H^*(\omega - \pi)e^{-j\omega} \quad , \quad g[k] = (-1)^{1-k} h[1-k]$$

$$G(\omega) = -H^*(\omega - \pi)e^{-j\omega} , \quad g[k] = (-1)^k h[1-k]$$

Operator: Down Sampling

$$y[n] = x[Mn]$$

$$Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X(\frac{\omega - 2\pi k}{M})$$

Proof:

$$Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} x[l]e^{-j(\omega-2\pi k)l/M} = \frac{1}{M}\sum_{l=0}^{N-1} x[l]e^{-j\omega l/M}\sum_{k=0}^{M-1} e^{j2\pi kl/M}$$

Only when $l = Mn$, $\sum_{k=0}^{M-1} e^{j2\pi kl/M} = M$

else $\sum_{k=0}^{M-1} e^{j2\pi kl/M} = 0$ (think Why ??)

Thus $Y(\omega) = \sum_{n=0}^{N/M-1} x[Mn]e^{j\omega n}$

$$y[n] = x[Mn]$$

Operator: Up Sampling

$$y[l] = \begin{cases} x[n], & l = Mn \\ 0, & others \end{cases}$$

$$Y(\omega) = x(M\omega)$$

Proof:

$$Y(\omega) = \sum_l y[l]e^{-j\omega l} = \sum_l x[n]e^{-j\omega Mn} = X(M\omega)$$

LAST REVIEW NOTE

1. Linear System

Finite vs. Infinite Duration Signals

Right-sided, Left-sided, and Two-sided Signals

The terms apply only to infinite duration signals.

Causal and Anti-Causal Signals

Zero phase, Linear phase and Nonlinear phase signals

Time Delay.

Linear System:

(1) Homogeneity.

(2) Additivity.

(3) Commutative.

Shift (Time) Invariance System.

Energy and Stability.

2. Convolution

Delta Function $\delta[n]$.

Standard Function $u[n]$.

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta(n-k)$$

Impulse response $h[n]$.

The signal that exits a system when a Delta Function (unit impulse) is the input.

$h[n] = S(\delta[n])$

Convolution $y[n] = h[n] * x[n]$

$$y[i] = \sum_{j=-\infty}^{\infty} h[j]x[i-j]$$

Properties of Delta function: $\delta[n]$ is identity for convolution.

$$\delta[n] * x[n] = x[n]$$

$$x[n] * k\delta[n] = kx[n]$$

$$x[n] * \delta[n+s] = x[n+s]$$

3. Fourier Representation

Fourier Response

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{jn\omega}d\omega$$

$$X(\omega) = X(\omega + 2\pi)$$

Discrete Fourier Transform (DFT)

$$x[n+N] = x[n]$$

$$X[k] = \sum_{i=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$$

$$X[k+N] = X[k]$$

Frequency Response of Sequence $h[n]$

$$H(\omega) = \sum_{n=-\infty}^{+\infty} h(n)e^{-j\omega n}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$|H(\omega)| = \sqrt{\operatorname{Im} H(\omega)^2 + \operatorname{Re} H(\omega)^2}$$ *Magnitude* of Transfer Function

$$Arg\left(H(\omega)\right) = \arctan\left(\frac{\operatorname{Im} H(\omega)}{\operatorname{Re} H(\omega)}\right)$$ *Phase* of Transfer Function

Ideal Low-pass Filter

$$H(\omega) = \begin{cases} 1, & |\omega| \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| \le \pi \end{cases}$$

$$h[n] = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi}, \quad h[2k] = 0.$$

Frequency Transform of Convolution

$$y[n] = h[n] * x[n], \quad \text{then } Y(\omega) = X(\omega)H(\omega)$$

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

$$y[n] = x[-n], \qquad Y(\omega) = X^*(\omega) \ \ or \ \ X(-\omega)$$

4. Fourier Pairs

Frequency Response

Delta Function

$x[n] = \delta[n]$, $\qquad X(\omega) = 1$

$x[n] = \delta[n - m]$, $\qquad X(\omega) = e^{-jm\omega}$

Square Function

Example 1:

$x[n] = 1, \quad n = 0, M-1$

$$X(\omega) = \frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}}$$

Example 2:

$x[n] = 1,\ |n| = 0, M$

$$X(\omega) = \frac{\sin\left(\frac{\omega(2M+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$y[n] = x[n - M],\ Y(\omega) = X(\omega)e^{-j\omega M}$

$y[n] = x[n]e^{jan},\ \ Y(\omega) = X(\omega - a)$

DFT

Delta Function

$X[k] = \mathrm{DFT}(\delta[n]) = 1$

$X[k] = \mathrm{DFT}(\delta[n - m]) = e^{-j2\pi km/N}$

$X[k] = \delta[k],\quad x[n] = \frac{1}{N}.$

$X[k] = \delta[k - M],\ x[n] = \frac{e^{j2\pi nM/N}}{N}.$

Square Function

$x[n] = 1,\ n = 0, M - 1$

$$X[k] = \frac{1 - e^{-j2\pi kM/N}}{1 - e^{-j2\pi k/N}} = e^{j\pi k(1-M)/N}\frac{\sin(\pi kM/N)}{\sin(\pi k/N)}$$

$x[n] = 1,\ \ n = 0, M$; and $x[n] = 1,\ n = N - M, N - 1.$

$$X[k] = \frac{\sin(\pi k(2M+1)/N)}{\sin(\pi k/N)}$$

Frequency Region:

$X[k] = 1,$ when $k = 0, M$;

$X[k] = 1,$ when $k = N - M, N - 1.$

Then $x[n] = \frac{1}{N}\frac{\sin(\pi n(2M+1)/N)}{\sin(\pi n/N)}$

$\text{DFT}(\ x[n - M]\ ) = X[k]e^{-j2\pi kM/N}$

$\text{IDFT}(\ X[k - M]\ ) = x[n]e^{j2\pi nM/N}$

5. FAST FOURIER TRANSFORM

Signal Length $N = 2^L$

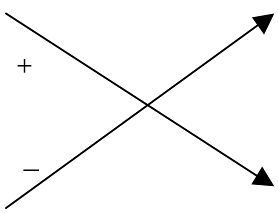
Let $x_1[r] = x[2r]$, $x_2[r] = x[2r + 1]$, $r = 0, N / 2 - 1.$

$$\boxed{X[k] = X_1[k] + X_2[k]e^{-j2\pi k/N},\ \ k = 0, N / 2 - 1.}$$

$$\boxed{X[k + N/2] = X_1[k] - X_2[k]e^{-j2\pi k/N},\ k = 0, N/2 - 1.}$$

Butterfly Algorithm

$X_1[k]$ $\qquad X_1[k] + X_2[k]e^{-j2\pi k/N}\ \ (X[k])$



$X_2[k]\ \boxed{e^{-j2\pi k/N}}$ $\qquad X_1[k] - X_2[k]e^{-j2\pi k/N}\ \ (X[k + N/2])$

One Butterfly: generates 2 Dots of DFT

Each Layer: $N/2$ Butterflies

Each Butterfly: 1 multiply + 2 add

Layer Number: $L = \log_2 N$

Calculation Efficiency:

FFT Cost = $(N/2) \times L \times$ (1 Multiply + 2 Add)

$= \frac{N}{2}\log_2 N$ Multiply + $N\log_2 N$ Add

DFT Cost = N^2 Multiply + $N(N - 1)$ Add

Filter

Finite Impulse Response (FIR) filter – Finite Convolution

Infinite Impulse Response (IIR) filter – Recursion

Low-pass, High-pass, Band-pass, and Band-reject filters

6. Wavelet

Operator: Down Sampling

$y[n] = x[Mn]$

$$Y(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\omega - 2\pi k}{M}\right)$$

Operator: Up Sampling

$$y[l] = \begin{cases} x[n], & l = Mn \\ 0, & others \end{cases}$$

$Y(\omega) = X(M\omega)$

$$\phi_{m,n}(x) = 2^{-m/2}\phi(2^{-m}x - n), \qquad \psi_{m,n}(x) = 2^{-m/2}\psi(2^{-m}x - n).$$

$$\phi(x/2) = 2\sum_n h[n]\phi(x-n), \qquad \psi(x/2) = 2\sum_n g[n]\phi(x-n).$$

$$\sum_n h[n] = 1, \qquad \sum_n g[n] = 0$$

Decomposition $C_m[n] = \sqrt{2}\sum_i C_{m-1}[i]h[i-2n]$, $D_m[n] = \sqrt{2}\sum_i C_{m-1}[i]g[i-2n]$.

Reconstruction $C_{m-1}[n] = \sqrt{2}\left(\sum_i C_m[i]h[n-2i] + \sum_l D_m[l]g[n-2l]\right)$.

For Computer Implementation

Decomposition $C_m[n] = \sum_i C_{m-1}[i]h[i-2n]$, $D_m[n] = \sum_i C_{m-1}[i]g[i-2n]$.

Reconstruction $C_{m-1}[n] = 2\times\left(\sum_i C_m[i]h[n-2i] + \sum_l D_m[l]g[n-2l]\right)$

Perfect Reconstruction Condition

$$|H(\omega)|^2 + |G(\omega)|^2 = 1$$
$$H(\omega)H^*(\omega-\pi) + G(\omega)G^*(\omega-\pi) = 0$$

Conjugate Quadrature Filter (CQF)

$$G(\omega) = H^*(\omega-\pi)\left[\pm e^{-j\omega(2M+1)}\right]$$

Then $H(\omega)H^*(\omega-\pi) + G(\omega)G^*(\omega-\pi) = 0$

$$|H(\omega)|^2 + |H(\omega-\pi)|^2 = 1$$
$$|G(\omega)|^2 + |G(\omega-\pi)|^2 = 1$$

CQF Example (Let $M = 0$)

$$G(\omega) = H^*(\omega-\pi)e^{-j\omega} \quad , \quad g[k] = (-1)^{1-k} h[1-k] .$$
$$G(\omega) = -H^*(\omega-\pi)e^{-j\omega} , \quad g[k] = (-1)^{k} h[1-k] .$$