Optimization Qualifying Exam (Dean)
Spring 2005

1) What is a **convex** set $C \in R^n$? What is an **epigraph** of a function $f : C \to R$. What is a **convex** function $f : C \to R$. What is a **quasi-convex** function $f : C \to R$.

2) Let C be a non-empty, open convex set in R^n and $f : C \to R$ be a differentiable function. Prove that f is convex on C if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in C.$$

3) Let f be a twice continuously differentiable function in R^n. What are the first-order necessary conditions of optimality? What are the second-order necessary conditions? What are the sufficient conditions?

4) Let $F : R^n \to R^n$ be continuously differentiable in the open convex set $D \subset R^n$, $x \in D$, and let F' be Lipschitz continuous at x in the neighborhood D. (We will use a vector norm, the induced matrix norm and the Lipschitz constant γ.) Then, for any $x + p \in D$, prove that

$$\|F(x + p) - F(x) - F'(x)p\| \leq \frac{\gamma}{2}\|p\|^2.$$

5) Write an algorithm for solving the unconstrained optimization problem:

$$\min_{x \in R^n} f(x)$$

where $f : R^n \to R$ is twice continuously differentiable. Use the BFGS method and an inexact line search method.

6) What is a **convex program**? Given a convex program, define the **dual program**.

7) Consider the nonlinear programming problem:

$$\begin{aligned} (NLP) \quad & \min_{x \in R^n} \; f(x) \\ \text{subject to } & h_i(x) = 0, \; i = 0, \ldots, p, \\ & g_i(x) \geq 0, \; i = 0, \ldots, m, \end{aligned}$$

where f, g_i, h_i are all twice continuously differentiable from R^n into R. If x^* satisfies the strong constraint qualification for problem (NLP) state the first order (KKT) and second order necessary conditions for problem (NLP). What are the sufficient conditions?

8) Consider the standard form of the linear program :

$$(LP) \quad \min_{x \in R^n} \; c^T x$$
$$\text{subject to } Ax = b, \quad x \geq 0,$$

where A is $m \times n$. State the dual problem. Prove the weak duality theorem. State the strong duality theorem.

9) Consider the problem:

$$(NEP) \; \min \; f(x)$$
$$\text{subject to } h_i(x) = 0, \; i = 0, \ldots, p,$$

where f, h_i are all twice continuously differentiable from R^n into R. Write an algorithm for solving (NEP) using an Augmented Lagrangian method. Write an algorithm for solving (NEP) using sequential quadratic programming.

10) Write a conjugate gradient algorithm for minimizing the quadratic $q(x) = \frac{1}{2}x^T Ax - b^T x$, where $A \in R^{n \times n}$ is symmetric positive definite and $x, b \in R^n$.