NUMERICAL ANALYSIS

Prelim Exam **2005 - Samples** **May, 2005**

This exam has 6 questions, for a total of 100 points.
Please answer the questions in the spaces provided on the question sheets.

Name and SSN: ______________________________

15 points

1. Suppose that we wish to solve $Ux = b$, where $b \in R^m$ and $U \in R^{m \times m}$, nonsingular and upper-triangular, are given, and $x \in R^m$ is unknown. Assume that we do this by the *back substitution algorithm*, i.e., solving for the components of x one after another, beginning with x_m and finishing with x_1.

   (a) Write down the back substitution algorithm.

   (b) Determine the exact numbers of additions, subtractions, multiplications, and divisions involving in the algorithm.

15 points

2. Let $A \in R^{m\times m}$ be nonsingular. Suppose that for each k with $1 \le k \le m$, the upper-left $k \times k$ block of A is nonsingular. Assume that that A is written in the block form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{11} is $n \times n$ and A_{22} is $(m-n)\times(m-n)$.

   (a) Verify the formula

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

   for "elimination" of the block A_{21}. The matrix $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is known as the *Schur complement* of A_{11} in A.

   (b) Suppose now that A_{21} is eliminated row by row by means of n *steps of Gaussian elimination without pivoting*:

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U = A, L = I
for k = 1 to n
   for j = k + 1 to m
      l_jk = u_jk / u_kk
      u_j,k:m = u_j,k:m − l_jk u_k,k:m
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   Show that the bottom-right $(m-n)\times(m-n)$ block of the result is again $A_{22} - A_{21}A_{11}^{-1}A_{12}$.

25 points

3. Consider the *conjugate gradient method* applied to a symmetric positive definite matrix problem $Ax = b$:

$x_0 = 0,\ r_0 = b,\ p_0 = r_0$	
for $n = 1, 2, \cdots$	
$\alpha_n = (r_{n-1}^T r_{n-1})/(p_{n-1}^T A p_{n-1})$	step length
$x_n = x_{n-1} + \alpha_n p_{n-1}$	approximate solution
$r_n = r_{n-1} - \alpha_n A p_{n-1}$	residual
$\beta_n = (r_n^T r_n)/(r_{n-1}^T r_{n-1})$	improvement this step
$p_n = r_n + \beta_n p_{n-1}$	search direction

(a) Show that the coefficients α_n and β_n in the conjugate gradient method can be written in the alternative (but less convenient) form

$$\alpha_n = \frac{p_{n-1}^T r_{n-1}}{p_{n-1}^T A p_{n-1}},$$
$$\beta_n = -\frac{p_{n-1}^T A r_n}{p_{n-1}^T A p_{n-1}}.$$

(b) Prove the three-terms recursive relation

$$A r_n = -\frac{1}{\alpha_{n+1}} r_{n+1} + \left(\frac{1}{\alpha_{n+1}} + \frac{\beta_n}{\alpha_n}\right) r_n - \frac{\beta_n}{\alpha_n} r_{n-1}$$

for the residual in the conjugate gradient method.

15 points

4. Suppose that we want to approximate the improper integral $I(f) = \int_0^\infty f(x)dx$ where $f(x) = cos^2(x)e^{-x}$. $I(f)$ can be approximated up to an absolute error equal to δ by first splitting $I(f)$ as $I(f) = I_1 + I_2$, where $I_1 = \int_0^c f(x)dx$ and $I_2 = \int_c^\infty f(x)dx$, then selecting c to guarantee that I_2 equals to $\delta/2$ and computing the corresponding I_1 up to an absolute error equals to $\delta/2$.

   (a) Let $\delta = 10^{-3}$. Find the constant $c > 0$ such that $I_2 \approx \delta/2$.

   (b) Estimate how many subintervals are needed if we use the composite trapezoidal formula for approximating I_1 up to an error $\delta/2$.

15 points 5. Let T_n be the Chebyshev polynomial of degree n.

(a) Show that the polynomial $p^*(x) = x^n - 2^{1-n}T_n$ is the polynomial of best approximation of x^n in the interval $[-1, 1]$ such that

$$\|x^n - p^*\|_\infty = \min_{p \in P_{n-1}} \|x^n - p\|_\infty,$$

where P_{n-1} is the vector space of polynomials of degree $n - 1$.

(b) Use the above result to prove the following minimax property of the Chebyshev polynomials

$$\|2^{1-n}T_n\|_\infty \leq \min_{p \in P_n^1} \|p\|_\infty,$$

where $P_n^1 = \{x^n\} + P_{n-1}$.

15 points 6. Consider the following family of linear multistep methods

$$u_{n+1} = \alpha u_n + \frac{h}{2}\left(2(1-\alpha)f_{n+1} + 3\alpha f_n - \alpha f_{n-1}\right)$$

where α is a real parameter.

(a) Analyze consistency and order of the methods as functions of α, determining the value α^* for which the resulting method has maximal order.

(b) Study the zero-stability of the method with $\alpha = \alpha^*$.

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.