Your Name:

Sample Preliminary Examination in Algebra

You have three hours to complete the exam. You cannot use any books or notes.

1. Let $\mathbf{G}$ and $\mathbf{H}$ be groups. Define that φ is a homomorphism from $\mathbf{G}$ to $\mathbf{H}$.
2. Define that $\mathbf{N}$ is a normal subgroup of $\mathbf{G}$ and define the factor group $\mathbf{G}/\mathbf{N}$. In particular, say what the elements of $\mathbf{G}/\mathbf{N}$ are and how multiplication, inverse and unit are defined.
3. Define $\ker(\varphi)$ for a homomorphism between groups and state the homomorphism theorem for groups.
4. List all subgroups of the additive group $\mathbb{Z}$ of integers and prove your answer.
5. Define cyclic groups and prove that every cyclic group is a homomorphic image of the additive group $\mathbb{Z}$ of integers.
6. Define that $\mathbf{I}$ is an ideal of the commutative ring $\mathbf{A}$ (with unit). Define the factor ring $\mathbf{A}/\mathbf{I}$ and prove that it is a field if and only if $\mathbf{I}$ is maximal.
7. (a) State Zorn's lemma.
   (b) Give an application of Zorn's lemma of your choice.
8. (a) Define that $\mathbf{D}$ is a p.i.d., that is a principal ideal domain.
   (b) Let $\mathbf{F}$ be a field. Sketch a proof that the polynomial ring $\mathbf{F}[x]$ is a p.i.d.
9. Let a and b be elements of a principal ideal domain $\mathbf{D}$.
   (a) Define: d is the greatest common divisor of a and b.
   (b) Prove that $\{xa + yb | x, y \in \mathbf{D}\}$ is the smallest ideal in $\mathbf{D}$ which contains a and b.
   (c) Prove that the greatest common divisor of a and b exists and that it is of the form $xa + yb$ for certain x and y in $\mathbf{D}$.
10. Let $(\mathbf{A}, +)$ be a commutative group and $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ be subgroups of $\mathbf{A}$. Define:
    (a) $\mathbf{A}$ is the sum of the subgroups $\mathbf{A}_i$.
    (b) $\mathbf{A}$ is the internal **direct** sum of the $\mathbf{A}_i$.
11. (a) State the structure theorem for finite abelian groups.
    (b) List up to isomorphism classes all abelian groups of order 144, e.g., in terms of direct sums of cyclic groups of prime power order.
12. Let $T : \mathbf{V} \to \mathbf{V}$ be a linear map on the vector space $\mathbf{V}$ over the field $\mathbf{F}$.
    (a) Define: $m_T(x)$ is the minimal polynomial of T.
    (b) Define: $c_T(x)$ is the characteristic polynomial of T.
    (c) State the *Caley-Hamilton* Theorem.
13. Assume that the characteristic polynomial for the linear map T on R^3 is $c_T(x) = (x-1)(x+1)^2$.
    (a) Find the minimal polynomial for T in case that T is cyclic.
    (b) Find the minimal polynomial in case that T has an eigen base.
    (c) Find all possible Jordan normal forms for T.