1. Suppose that $f : [0, L] \to \mathbb{R}$ is a C^2-function with $f'(0) < 0$, $f''(0) > 0$ on $(0, L)$ and that you have explicit formulae for $f(x)$ and $f'(x)$ that can be evaluated.
   (a) Describe a simple test for verifying that the minimizer of f on $[0, L]$ is L.
   (b) Suppose that the test in (a) says that the minimizer is not at L. Describe an algorithm, including a stopping criterion, for finding an interval of length less than a preassigned ϵ that contains a local minimizer.

2. Suppose that $f : \mathbb{R}^N \to \mathbb{R}$ is a C^2-function and one wishes to minimize the function on a closed ball B of radius R and center $x^{(0)} \in \mathbb{R}^N$.
   (a) Does this function f always have a unique minimizer on the ball B?
   (b) Suppose that a point $\tilde{x}$ is a local minimizer of f and $|\tilde{x} - x^{(0)}| < R$. What extremality conditions hold at $\tilde{x}$?
   (c) Suppose that a point $\tilde{x}$ is a local minimizer of f on B and $|\tilde{x} - x^{(0)}| = R$. What extremality conditions hold at $\tilde{x}$?
   (d) Give the explicit forms of your answers to (b) and (c) when $f(x) := x^T Ax - b^T x$ is a quadratic function. Here A is an $N \times N$ symmetric matrix and $b \in \mathbb{R}^N$.

3. Consider the problem of extremizing a C^1-convex function $f : \overline{B}_1 \to \mathbb{R}$ on the closed unit ball $\overline{B}_1 \subset \mathbb{R}^N$.
   (a) Give reasons why there is at least one minimizer and one maximizer of f on $\overline{B}_1$.
   (b) Describe the usual Lagrangian function for finding the minimizers of this constrained problem.
   (c) Write out the equations and inequalities that hold at a minimizer of this problem.
   (d) What changes in your answers to parts (b) and (c) if minimizer is replaced by maximizer?
   (e) Describe a barrier function that will convert the minimization problem in to an unconstrained optimization problem. Describe how you would approximate the minimizers of f on $\overline{B}_1$ by using this barrier function.

4. Consider the optimization problem

$$\begin{aligned} &\text{minimize } x^2 + 1 \\ &\text{subject to } (x-2)(x-4) \le 0, \end{aligned}$$

   with variable $x \in \mathbb{R}$.
   (a) Give the feasible set, the optimal value, and the optimal solution.
   (b) Plot the objective $x^2 + 1$ versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x, \lambda)$ versus x for a few positive values of λ. Verify the lower bound property ($p^* \ge \inf_x L(x, \lambda)$ for $\lambda \ge 0$). Derive and sketch the Lagrange dual function g.
   (c) State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution λ^*. Does strong duality hold?
   (d) Let $p^*(u)$ denote the optimal value of the problem

$$\begin{aligned} &\text{minimize } x^2 + 1 \\ &\text{subject to } (x-2)(x-4) \le u, \end{aligned}$$

   as a function of the parameter u. Plot $p^*(u)$. Verify that $dp^*(0)/du = -\lambda^*$.

5. (*Simplex Method*)

Consider the linear program

$$\begin{aligned} \text{maximize } & x_7 \\ \text{subject to } & x_1 + 4x_2 + x_3 = 1400 \\ & 2x_1 + 3x_2 + x_4 = 2000 \\ & x_1 + 12x_2 + x_5 = 3600 \\ & 2x_1 + x_6 = 1800 \\ & 40x_1 + 200x_2 - x_7 = 0 \\ & x_i \geq 0,\ 1 \leq i \leq 6. \end{aligned}$$

Use the simplex method with $J_0 = \{3, 4, 5, 6, 7\}$ as a feasible start basis to compute an optimal solution.

6. (*Dynamic Programming*)

The Department of Mathematics needs to have a certain copying machine over the next five year period. Each new machine costs $c = \$1000$. The cost m_i, $1 \leq i \leq 3$, of maintaining the machine during its i^{th} year of operation is $m_1 = \$60$, $m_2 = \$80$, $m_3 = \$120$. A machine may be kept up to three years before being traded in. The trade in value s_i, $1 \leq i \leq 3$, after i years is $s_1 = \$800$, $s_2 = \$600$, $s_3 = \$500$. The chairman of the department wants to minimize costs over the five year period.

The problem can be solved by dynamic programming: Let the stages correspond to each year. The state is the age of the machine for that year. The decision is whether to keep the machine or trade it in for a new one. Let $f_t(j)$ denote the minimum cost incurred from time $0 \leq t \leq 5$ to time 5, given the machine is j years old at time t.

(a) Show that the application of Bellman's optimality principle leads to the following backward dynamic programming algorithm:

$$\begin{aligned} f_5(j) &= -s_j, \\ f_t(3) &= -s_3 + c + m_1 + f_{t+1}(1), \\ f_t(j) &= \min\{-s_j + c + m_1 + f_{t+1}(1),\ m_{j+1} + f_{t+1}(j+1) \mid j \in \{1,2\}\}, \\ f_0(0) &= c + m_1 + f_1(1), \end{aligned}$$

with $f_0(0)$ being the minimum cost.

(b) Perform the backward dynamic programming algorithm and complete the following tables by the results of your computations.

Stage 5:

Age j	$f_5(j)$
1	
2	
3	

Stage t $(t = 4, 3, 2, 1)$:

Age j	Trade	Keep	$f_5(j)$	Decision
1				
.				
$\min\{3, t\}$				

Stage 0:

Age j	Keep	$f_0(j)$
0		

In columns 'Trade' and 'Keep' fill in the trading and keeping costs, respectively, and in column 'Decision' fill in 'Keep' or 'Trade' based on the decision whether to keep the machine or to trade it in. If the keeping cost equals the trading cost fill in 'Keep or Trade'.

7. (*Pontrjagin's Minimum Principle*)

   Given two points $A = (0, a)$ and $B = (T, b)$ in the plane with $T > 0$ and $a, b \in \mathbb{R}$, the problem is to find a curve $y \in C^1([0, T])$ of minimum length that connects A and B.

   (a) Show that the problem can be formulated as the minimization problem

   $$\begin{aligned} &\text{minimize} \int_0^T \sqrt{1 + \left(\frac{dy(t)}{dt}\right)^2}\, dt, \\ &\text{subject to } y(0) = a,\ y(T) = b. \end{aligned}$$

   (b) Reformulate the problem from (a) as a continuous-time optimal control problem and solve this problem by Pontrjagin's minimum principle.

8. Given m distinct vectors $(a_1, b_1), \ldots, (a_m, b_m)$ in $\mathbb{R}^2$, define

   $$\varphi(x) := \max_{1 \le j \le m} (a_j x + b_j).$$

   (a) Is the function $\varphi \ldots$
      i. continuous on $\mathbb{R}$?
      ii. a differentiable function on $\mathbb{R}$?
      iii. a convex function on $\mathbb{R}$?

   (b) Take $m = 2$ and sketch a typical graph of a function φ. Label the graph and indicate any conditions for its validity.

   Give conditions on the data for φ to have a finite minimum value on $\mathbb{R}$. When these conditions hold, find the minimizer of φ explicitly and evaluate the minimum value.

9. Suppose A in an $M \times N$ matrix with $M < N$ such that A has rank M. Suppose $b \in \mathbb{R}^M$. What can you say about the set W of all solutions of the equation $Ax = b$? (State as many distinct properties as you can.) In particular, is the set W non-empty? Give reasons for your answer.

   Consider the problem of minimizing the 2-norm function $d(x) := \|x\|_2^2$ subject to the constraint $Ax = b$. What are the equations that hold at a minimizer of this problem? Outline an algorithm for finding such minimizers.

10. Suppose $f : [0, L] \to \mathbb{R}$ is a C^1-function with $f'(0) < 0$, $f'(L) > 0$ on $(0, L)$, and suppose you have explicit formulae for $f(x)$ and $f'(x)$ that can be evaluated.

    (a) Can you guarantee that this function has a minimizer on $[0, L]$? Give reasons for your answer.

    (b) Outline an algorithm that is guaranteed to find an interval of length less than a preassigned ϵ that contains a local minimizer of f on $(0, L)$. What is the stopping criterion for this algorithm?

11. Show that the following functions are convex.

    (a) $f(x) = e^x - 1$ on $\mathbb{R}$

    (b) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$

    (c) $f(x_1, x_2) = x_1^2 / x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$

    (d) $f(x) = -\left(\sum_{i=1}^n x_i^p\right)^{1/p}$ with $p < 1$, $p \neq 0$, and $\mathrm{dom}(f) = \mathbb{R}^n_{++}$

12. Formulate the following problems as LPs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP.

    (a) Minimize $\|Ax - b\|_\infty$ (ℓ_∞-norm approximation).

    (b) Minimize $\|Ax - b\|_1$ (ℓ_1-norm approximation).

(c) Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$.

(d) Minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$.

13. (*Calculus of Variations*)

Consider the problem of finding a smooth curve of the form $y(x)$ which will provide the shortest distance from the origin to the parabola given by $y = x^2 - 1$.

(a) What is the Euler-Lagrange equation for this problem?

(b) Show that there are precisely two points (t, y) and $(\tilde{t}, \tilde{y})$ on the parabola which satisfy the transversal condition.

(c) Find the associated curves which represent the possible extremals.

(d) What happens if the parabola is replaced by the circle $x^2 + y^2 = 1$?

14. (*Continuous time optimal control*)

Consider the linear-quadratic optimal control problem

$$\text{minimize } J(y, u) := \frac{1}{2} q(y(T))^2 + \frac{1}{2} \int_0^T (u(t))^2 \, dt$$

subject to the one-dimensional linear system

$$\begin{aligned} \dot{y}(t) &= ay(t) + bu(t), \quad t \in [0, T] \\ y(0) &= y_0, \end{aligned}$$

where $a, b \in \mathbb{R}$, $q > 0$, $T > 0$, and $y_0 \in \mathbb{R}$ are given.

Show that the optimal solution $y^*(t)$, $u^*(t)$, $0 \leq t \leq T$, is given by

$$\begin{aligned} y^*(t) &= y_0 \exp(at) + \frac{b^2 c}{2a} (\exp(-at) - \exp(at)) \\ u^*(t) &= -bc \exp(-at). \end{aligned}$$

Compute the constant c.

[Hint: Verify that the conditions of Pontrjagin's minimum principle are satisfied. In particular, compute the Hamiltonian and the adjoint equation. Why are the conditions sufficient in this case?]

15. (*Simplex Method*)

Consider the linear program

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & -3x_1 - x_2 - 3x_3 \\ \text{subject to} \quad & \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \quad x \geq 0. \end{aligned}$$

By introducing slack variables x_i, $4 \leq i \leq 6$, transform the problem to standard form max x_6 subject to $Ax = b$, $x_i \geq 0$, $1 \leq i \leq 5$, and solve it by the simplex method using $J_0 = \{4, 5, 6\}$ as a feasible start basis.

16. (*Nonlinear Programming*)

Assume $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^p$, $h : \mathbb{R}^n \to \mathbb{R}^m$, and consider the nonlinear program (NLP):

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{over} \quad & x \in \mathbb{R}^n \\ \text{subject to} \quad & h(x) = 0, \\ & g(x) \leq 0. \end{aligned}$$

(a) What does the Linear Independence Constraint Qualification (LICQ) say?

(b) Specify the second order sufficient optimality conditions.

(c) Assume that the NLP is solved by Sequential Quadratic Programming (SQP) with an approximation B_k of the Hessian of the associated Lagrangian at the k^{th} iteration step. What does the Quadratic Programming subproblem look like?

17. Let A be an $M \times N$ matrix, $b \in \mathbb{R}^M$, and consider the least squares function $f(x) := \|Ax - b\|^2$.

(a) Evaluate $\nabla f(x)$ and $D^2 f(x)$.

(b) Does this problem always have a minimizer? **Y** or **N**?

(c) Find the equations that any such minimizers must satisfy.

(d) Give a criterion on the data that guarantees that there is a unique minimizer of f on $\mathbb{R}^N$.

18. Consider the problem of minimizing the quadratic function

$$f(x) := b^T x + x^T Ax, \quad \text{subject to } \|x\|_2 \leq \Delta.$$

Here A is an $N \times N$ symmetric matrix, $b \in \mathbb{R}^N$, and $\Delta > 0$.

(a) Does this problem always have a minimizer? **Y** or **N**?

(b) Give reasons for your answer to (a).

(c) Give a careful description of the equations that must hold at a minimizer of f on this ball.

(d) Give the criterion for a unit vector $d \in \mathbb{R}^N$ to be a descent direction for f at the origin.

19. Suppose A is an $N \times N$ real symmetric positive definite matrix and $b \in \mathbb{R}^N$. Consider the problem of minimizing the function

$$E(x) := x^T Ax - 2b^T x \quad \text{ on } \mathbb{R}^N.$$

(a) What equations does the minimizer of this problem satisfy?

(b) Can you guarantee that there is a minimizer of this problem? Give reasons why or why not.

(c) Describe (outline) the conjugate gradient algorithm for minimizing E on $\mathbb{R}^N$.

(d) What is a good stopping criterion for the algorithm you gave in (c)?

(e) Does the value of E strictly decrease at each step in your algorithm provided the stopping criterion does not hold? **Y** or **N**? Give reasons for your answer.

20. Consider the problem of minimizing a C^1-function f defined on a neighborhood of the line segment $\Gamma := \{(x_1, x_2) : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 = 1\}$.

(a) Suppose that a point (a_1, a_2) is a local minimizer of f on this line segment with $a_1 \cdot a_2 \neq 0$. What equations will hold at this local minimizer?

(b) Consider $f(x_1, x_2) := \frac{x_2}{2} - x_1 x_2$. Evaluate the gradient and the Hessian of this function f.

(c) Is this function convex on $\mathbb{R}^2$? **Y** or **N**?

(d) Find the minimizers and maximizers of this function on the line segment Γ.

21. Show that the following sets are convex:

(a) A wedge, i.e. $\{x \in \mathbb{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.

(b) The set of points closer to a given point than a given set, i.e.

$$\{x \in \mathbb{R}^n \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $x_0 \in \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$.

(c) The set $\{x \in \mathbb{R}^n \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

(d) The set of points whose distance to the point $a \in \mathbb{R}^n$ does not exceed a fixed fraction θ of the distance to the point $b \in \mathbb{R}^n$, i.e. the set $\{x \in \mathbb{R}^n \mid \|x-a\|_2 \le \theta\|x-b\|_2\}$. You can assume $a \neq b$ and $0 \le \theta \le 1$.

22. Consider the problem of minimizing a quadratic function:

$$\text{minimize} \quad f(x) = (1/2)x^T P x + q^T x + r,$$

where $P \in S^n$ (but we do not assume that $P \geq 0$).

(a) Show that if $P \ngeq 0$, i.e. the objective function f is not convex, then the problem is unbounded below.

(b) Now suppose that $P \geq 0$ (so the objective function is convex), but the optimality condition $Px^* = -q$ does not have a solution. Show that the problem is unbounded below.

23. (*Quadratic Programming Problem*)

Consider the QP

$$\begin{aligned} \text{minimize} \quad & 3x_1^2 + 2x_1x_2 + x_1x_3 + 2.5x_2^2 + 2x_2x_3 + 2x_3^2 - 8x_1 - 3x_2 - 3x_3 \\ \text{over} \quad & x = (x_1, x_2)^T \in \mathcal{D}^2 \\ \text{subject to} \quad & x_1 + x_3 = 3, \\ & x_2 + x_3 = 0. \end{aligned}$$

The associated KKT system is given as follows:

$$\begin{pmatrix} B & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix}.$$

(a) Identify the matrices $B \in \mathcal{D}^{3\times 3}$, $A \in \mathcal{D}^{2\times 3}$, and the vectors $b \in \mathcal{D}^3$, $c \in \mathcal{D}^2$ of the KKT system and compute its solution x^*, λ^*.

(b) Solve the QP problem by the null-space approach. In particular, specify the null-space basis matrix $Z \in \mathcal{D}^{3\times 1}$, i.e. $AZ = 0$, and a matrix $Y \in \mathcal{D}^{3\times 2}$ such that the matrix $[Y|Z] \in \mathcal{D}^{3\times 3}$ is regular. Compute w_Y, w_Z, and λ^* as the solutions of $AYw_Y = c$, $Z^T BZw_Z = Z^T b - Z^T BYw_Y$, and $(AY)^T \lambda^* = Y^T b - Y^T Bx^*$, where $x^* = Yw_Y + Zw_Z$. (Hint: Choose Y such that $AY = I$, where I is the 2×2 unit matrix.)

24. (*Optimal Control Problem*)

A young investor has earned in the stock market a large amount of money S and plans to spend it so as to maximize his enjoyment through the rest of his life without working. He estimates that he will live exactly T more years and that his capital $x(t)$ should be reduced to zero at time T, i.e. $x(T) = 0$. Also he models the evolution of his capital by the differential equation

$$\frac{dx(t)}{dt} = \alpha x(t) - u(t),$$

where $x(0) = S$ is his initial capital, $\alpha > 0$ is a given interest rate, and $u(t) \geq 0$ is his rate of expenditure. The total enjoyment he will obtain is given by

$$\int_0^T e^{-\beta t} \sqrt{u(t)}\, dt.$$

Here β is some positive scalar which serves to discount future enjoyment. Find the optimal $\{u(t) \mid t \in [0, T]\}$.

25. Consider the linear programming (LP) problem of minimizing

$$f(x) := c^T x \quad \text{subject to } Ax = b \text{ and } x \geq 0.$$

Here A is an $M \times N$ matrix with $1 \leq M < N$.

(a) Describe a Lagrangian function for this system and indicate why the function you have given is a Lagrangian for this problem.

(b) Give the extremality conditions that must hold at a local minimizer of this problem.

(c) Describe a dual linear programming problem to this problem.