Grad Macro I - Fall 2007
Final Exam Answers

1 Problem 1

A) Rewrite the production function as

$$y = k^{\alpha-\beta} g^{\beta} \tag{1}$$

so to have endogenous growth we must have that $\alpha \geq 1$. This holds regardless of what beta is.

B) So assume now that $\alpha = 1$. and we have the following accumulation equations (note that lot's of people overlooked that g has an accumulation process as well. READ THE PROBLEM!)

$$\begin{aligned} \dot{k} &= s(1-T) k^{1-\beta} g^{\beta} - \delta k & (2) \\ \dot{g} &= T k^{1-\beta} g^{\beta} - \delta g & (3) \end{aligned}$$

and in steady state we would have to have that

$$\frac{\dot{k}}{k} = \frac{\dot{g}}{g} \tag{4}$$

and when you set this up you get

$$s(1-T) k^{-\beta} g^{\beta} - \delta = T k^{1-\beta} g^{\beta-1} - \delta \tag{5}$$

and this solves to

$$\left(\frac{k}{g}\right)^{*} = \frac{s(1-T)}{T}. \tag{6}$$

C) Growth of output per capita at steady state is

$$\begin{aligned} \frac{\dot{y}}{y} &= (1-\beta) \frac{\dot{k}}{k} + \beta \frac{\dot{g}}{g} & (7) \\ &= \frac{\dot{k}}{k} & (8) \end{aligned}$$

since $\frac{\dot{k}}{k} = \frac{\dot{g}}{g}$. Plug in the steady state ratio of k/g into the equation for $\dot{k}/k$ and you get

$$\frac{\dot{y}}{y} = s^{1-\beta} (1-T)^{1-\beta} T^{\beta} - \delta. \tag{9}$$

Maximize this growth rate over T and you'll get

$$T^{*} = \beta. \tag{10}$$

2 Problem 2

A) Write down the Euler equation that must hold between any two adjacent periods

$$\frac{U'(C_t)}{U'(C_{t+1})} = \frac{1+r_t}{1+\theta} \tag{11}$$

and plugging in for the utility function gives

$$\frac{\alpha C_t^{\alpha-1} G_t^{\beta}}{\alpha C_{t+1}^{\alpha-1} G_{t+1}^{\beta}} = \frac{1+r_t}{1+\theta} \tag{12}$$

and rearrange this for r yields

$$1 + r_t = (1+\theta)\left(\frac{G_t}{G_{t+1}}\right)^{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\alpha}. \tag{13}$$

Notice that from the problem we know that $C_t = C_{t+1} = 1$ so we can rewrite this as

$$1 + r_t = (1+\theta)\left(\frac{G_t}{G_{t+1}}\right)^{\beta}. \tag{14}$$

For odd periods, $G_t = 1$, $G_{t+1} = 2$, so

$$1 + r_{ODD} = (1+\theta)\left(\frac{1}{2}\right)^{\beta} \tag{15}$$

while in even periods we have (just reverse the periods)

$$1 + r_{EVEN} = (1+\theta)\left(\frac{1}{2}\right)^{\beta}. \tag{16}$$

3 Problem 3

A) The utility function for this person is

$$U = \ln C_1 + \ln C_2 + \beta \ln n \tag{17}$$

and the budget constraint is

$$C_1 + \frac{C_2}{1+r} = w\left(1 - \theta n\right). \tag{18}$$

Solve the budget for C_1 and plug it into the utility function. Take FOC with respect to C_2 and n. This gives us

$$\frac{-1/(1+r)}{w(1-\theta n) - C_2/(1+r)} + \frac{1}{C_2} = 0 \tag{19}$$

$$\frac{-\theta w}{w(1-\theta n) - C_2/(1+r)} + \frac{\beta}{n} = 0 \tag{20}$$

you can solve these two conditions together to find that

$$n^* = \frac{\beta}{(2+\beta)\,\theta} \tag{21}$$

$$S^* = w\frac{(1+\beta)}{(2+\beta)}. \tag{22}$$

B) Capital accumulation is

$$k_{t+1} = \frac{S_t}{n_t} \tag{23}$$

$$= \frac{w_t\frac{(1+\beta)}{(2+\beta)}}{\frac{\beta}{(2+\beta)\theta}} \tag{24}$$

$$= w_t\frac{1+\beta}{\beta}\theta \tag{25}$$

and wages must be $w_t = (1-\alpha)\,k_t^{\alpha}$ due to our production function. This gives us

$$k_{t+1} = \frac{1+\beta}{\beta}\theta\,(1-\alpha)\,k_t^{\alpha} \tag{26}$$

and this can be graphed. There is a steady state where the k_{t+1} line crosses the 45 degree line. If the cost of children goes up (θ increases) then k_{t+1} is higher for any given k_t, and the steady state must be higher. Why? Because we have fewer kids, and this means more capital per person.

C) Now if $\theta = k_t^{\delta}$ then

$$k_{t+1} = \frac{1+\beta}{\beta}\,(1-\alpha)\,k_t^{\alpha+\delta} \tag{27}$$

and I told you in the problem that $\alpha + \delta > 1$. If you graph this, you have a convex curve (not concave as before), meaning that it starts out below the 45 degree line and then grows exponentially after that. This means there is a steady state - at the point where the k_{t+1} curve crosses the 45 degree line, but not this steady state is unstable. Can we have endogenous growth? Yes, if we have initial capital high enough, so that $k_0 > k^*$, then capital will grow forever, and so will income.

4 Problem 4

A) We have

$$U = \ln c_1 + p\ln c_2 \tag{28}$$

such that

$$c_1 + c_2 = w_1 + w_2 \tag{29}$$

and if we solve this we get a FOC of

$$\frac{-1}{w_1 + w_2 - c_2} + \frac{p}{c_2} = 0 \tag{30}$$

which we can solve with the budget constraint to find

$$S = \frac{p}{1+p} w_1 - \frac{w_2}{1+p} \tag{31}$$

so that we have

$$\frac{\partial S}{\partial p} = \frac{1}{(1+p)^2} (w_1 + w_2) > 0 \tag{32}$$

or savings always goes up when the probability of being alive tomorrow increases.

B) When are savings negative? When $S < 0$ or

$$\frac{p}{1+p} w_1 - \frac{w_2}{1+p} < 0 \tag{33}$$

and this means that savings are negative when

$$p < \frac{w_2}{w_1}. \tag{34}$$

In other words, if my future income is really high relative to the chance of being alive in the second period, then I borrow (have negative savings).

C) If you cannot borrow at all, then what happens? Well, if $p > w_2/w_1$, then it doesn't matter to you, as you are saving anyway. If $p < w_2/w_1$, then you'd like to borrow, but you can't. You'll try to get as close to your optimal point as possible, which means you consume everything you can in the first period. So $c_1 = w_1$ and by default $c_2 = w_2$.

D) Take utility under the borrowing constraint first. Let's assume that $p < w_2/w_1$, so that you would like to borrow. Your utility is

$$U = \ln w_1 + p \ln w_2 \tag{35}$$

and it must be that U increases with p. In other words, a higher chance of living longer increases utility.

Now, what if you are not constrained? Now your utility is

$$U = \ln \frac{1}{1+p} (w_1 + w_2) + p \ln \frac{p}{1+p} (w_1 + w_2) \tag{36}$$

$$= \ln (w_1 + w_2) + p \ln (w_1 + w_2) - \ln (1+p) + p \ln p - p \ln (1+p) \tag{37}$$

Take the derivative with respect to p

$$\frac{dU}{dp} = \ln (w_1 + w_2) - \frac{1}{1+p} + \ln p - \ln (1+p) - \frac{p}{1+p} \tag{38}$$

$$= \ln (w_1 + w_2) + \ln p - \ln (1+p) \tag{39}$$

and this is negative if

$$\ln(w_1 + w_2) + \ln p - \ln(1 + p) < 0 \tag{40}$$

$$\ln(w_1 + w_2) < \ln(1 + p) - \ln p \tag{41}$$

$$w_1 + w_2 < \frac{1 + p}{p}. \tag{42}$$

So if wages are low enough, then yes it is possible for the chance of living longer to lower utility. Note that as p goes to zero, then the RHS goes to infinity and it must be that increases in p make you worse off. So when life expectancy is low, we could guess that people are actually made worse off by improvements to their longevity.