1 Problem 10

Here we are modifying the expanding varieties model. Production is

$$Y_i = L_i^{1-\alpha} \sum_{j=1}^{N} X_{ij}^{\alpha} \tag{1}$$

and the intermediate producers of X are still monopolists, while the final goods sector is perfectly competitive. Now, though, starting a new variety is not just fixed at the level d. Now, you have to hire people to "invent" the new variety, and the cost of this labor effort is equal to

$$w\frac{\eta}{N}. \tag{2}$$

Let's start by looking at aggregate production. We know that for each final goods firm, they will still use an equal amount of each intermediate good X, so their production is

$$Y_i = L_i^{1-\alpha} N X_i^{\alpha}. \tag{3}$$

Each final goods firm is still profit maximizing, and their profits are

$$\pi_i = Y_i - wL_i - \sum_{j=1}^{N} P_j X_{ij} \tag{4}$$

and by taking FOC and rearranging we get the demand for good X_{ij} of

$$X_{ij} = L_i \left(\frac{\alpha}{P_j}\right)^{1/1-\alpha} \tag{5}$$

and therefore economy-wide demand for goods is

$$X_j = \sum_i X_{ij} = \left(\frac{\alpha}{P_j}\right)^{1/1-\alpha} \sum_i L_i = \left(\frac{\alpha}{P_j}\right)^{1/1-\alpha} (L - L_R) \tag{6}$$

where total labor is equal to only $(L - L_R)$ rather than L, the total population. The monopolistically competitive firms will still set the profit maximizing price of

$$P_j = \frac{1}{\alpha} \tag{7}$$

so that the total amount of each intermediate good is

$$X_j = (L - L_R)\,\alpha^{2/1-\alpha} \tag{8}$$

So what is the value of an invention for a monopolistic intermediate producer? Similar to before it is

$$V = \left(\frac{1}{\alpha} - 1\right)(L - L_R)\,\alpha^{2/1-\alpha}\frac{1}{r}. \tag{9}$$

We need to equate V to the cost of inventing, $w\frac{\eta}{N}$, but that depends on the wage rate. What will the wage be? Well, let's look at aggregate output, which is

$$\begin{aligned} Y &= \sum_i Y_i = \sum_i L_i^{1-\alpha} N X_i^{\alpha} && (10) \\ &= N \sum_i L_i^{1-\alpha} \left((L - L_R)\, \alpha^{2/1-\alpha} \right)^{\alpha} && (11) \\ &= N \left((L - L_R)\, \alpha^{2/1-\alpha} \right)^{\alpha} \sum_i L_i^{1-\alpha} && (12) \\ &= N \left((L - L_R)\, \alpha^{2/1-\alpha} \right)^{\alpha} (L - L_R)^{1-\alpha} && (13) \\ &= N (L - L_R)\, \alpha^{2\alpha/1-\alpha} && (14) \end{aligned}$$

This means the wage, which is the marginal product of labor, is

$$w = N\alpha^{2\alpha/1-\alpha} \tag{15}$$

or is increasing in N. Now, back to the monopolistic producers, who set $V = w\frac{\eta}{N}$

$$\begin{aligned} \left(\frac{1}{\alpha} - 1\right)(L - L_R)\, \alpha^{2/1-\alpha} \frac{1}{r} &= N\alpha^{2\alpha/1-\alpha} \frac{\eta}{N} && (16) \\ \left(\frac{1}{\alpha} - 1\right) \frac{(L - L_R)}{\eta} &= r && (17) \end{aligned}$$

and this pins down the interest rate as a function of the number of people doing research.

Now, we know that a single invention requires $\frac{\eta}{N}$ units of labor. The change in N is therefore given by

$$\dot{N} = \frac{L_R}{\eta/N} \tag{18}$$

which can be written as

$$\frac{\dot{N}}{N} = \frac{L_R}{\eta} \tag{19}$$

which tells us that the number of varieties is increasing in the number of researchers, kind of like the expanding quality model.

Now, we know from the consumption maximization area that

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}(r - \theta) \tag{20}$$

and we saw in class that C and Y and N have to grow at the same rate. So we have that

$$\begin{aligned} \frac{\dot{N}}{N} &= \frac{L_R}{\eta} = \frac{1}{\sigma}(r - \theta) && (21) \\ \frac{L_R}{\eta} &= \frac{1}{\sigma}\left(\left(\frac{1}{\alpha} - 1\right) \frac{(L - L_R)}{\eta} - \theta \right) && (22) \end{aligned}$$

which you can solve for

$$L_R = \frac{\alpha L - \eta\theta}{\alpha + \sigma} \tag{23}$$

and therefore

$$r = \alpha \frac{\sigma L/\eta + \theta}{\alpha + \sigma} \tag{24}$$

and the growth rate of y, N, and C is

$$\frac{\dot{y}}{y} = \frac{\alpha L/\eta - \theta}{\sigma + \alpha} \tag{25}$$

2 Problem 11

A) You have to choose x, the number of people who work, and n, the number of people who do research. What is welfare?

$$\begin{aligned} U &= \int_0^\infty e^{-rt} E(y_t) dt && (26) \\ &= \int_0^\infty e^{-rt} \left(\sum_{t=0}^{\infty} \Pi(k,t) A_k x^\alpha \right) dt && (27) \end{aligned}$$

where we used the idea that

$$E(y_t) = \sum_{k=0}^{\infty} \Pi(k,t) A_k x^\alpha \tag{28}$$

or the expected value of output at time t is just the expected value of the number of innovations that have occurred up to the time t. Using the fact I gave you about $\Pi(k,t)$ we have that

$$\begin{aligned} U &= \int_0^\infty e^{-rt} \left(\sum_{k=0}^{\infty} \Pi(k,t) A_0 \gamma^k x^\alpha \right) dt && (29) \\ &= A_0 x^\alpha \left(\int_0^\infty e^{-rt} e^{-\lambda n t(1-\gamma)} \sum_{k=0}^{\infty} \frac{(\lambda n \gamma t)^k}{k!} e^{-\lambda n \gamma t} dt \right) && (30) \\ &= A_0 x^\alpha \left(\int_0^\infty e^{-(r + \lambda n - \lambda n \gamma)t} dt \right) && (31) \end{aligned}$$

which follows because

$$\sum_{k=0}^{\infty} \frac{(\lambda n \gamma t)^k}{k!} e^{-\lambda n \gamma t} = 1 \tag{32}$$

as it is just the pdf of a probability distribution. That means that we have

$$A_0 x^\alpha \left(\int_0^\infty e^{-(r+\lambda n - \lambda n \gamma)t} dt \right) \tag{33}$$

$$= \frac{A_0 x^\alpha}{r + \lambda n - \lambda n \gamma} \tag{34}$$

$$= \frac{A_0 \left(L - n\right)^\alpha}{r + \lambda n - \lambda n \gamma} \tag{35}$$

and maximizing utility over n yields the following FOC

$$1 = \frac{\lambda \left(\gamma - 1\right) \left(1/\alpha\right) \left(L - n^*\right)}{r + \lambda n^* - \lambda n^* \gamma}. \tag{36}$$

Now compare this to the decentralized condition that we found in class

$$1 = \frac{\lambda \gamma \frac{1-\alpha}{\alpha} \left(L - n\right)}{r + \lambda n} \tag{37}$$

and there are three differences to note. First, the social planner has a lower discount rate: $r + \lambda n^* - \lambda n^* \gamma < r + \lambda n$. Why? Because the social planner takes into account the positive effects of a new innovation, while the individual is only concerned with their own profits. This would tend to make growth be higher in the social planners case.

Second, note that in the numerator of the social planner we have $1/\alpha$ while the individuals problem has $\left(1 - \alpha\right)/\alpha$, and thus the individuals numerator is smaller. This arises because the individual, while a monopolist, can only extract $1 - \alpha$ of final output, while the social planner has all of output at his disposal. This effect as well will make growth higher for the social planner.

Finally, note that in the numerator of the social planner solution, we have $(\gamma - 1)$ while the individual has γ. What is happening here? The social planner does not assign the full value γ to an innovation, because he realizes that the innovation has stolen someone elses monopoly, whereas the individual doesn't care. So in this case, the social planner will actually pursue lower growth (that is, will choose a lower level of n^*).

Without doing a lot of algebra, the key to getting individuals to have growth be "too high" is to have this $(\gamma - 1) < \gamma$ effect dominate, and then the social planners solution (which maximizes welfare) have lower growth than the individual outcome.