We numerically study the displacement flow of two iso-viscous Newtonian fluids in an inclined two-dimensional channel, formed by two parallel plates. The results are complementary to our previous studies on displacement flows in pipes and channels. The heavier displacing fluid moves the lighter displaced fluid in the downward direction. Three dimensionless groups largely describe these flows: the densimetric Froude number ($Fr$), the Reynolds number ($Re$), and the duct inclination ($\beta$). As a first order approximation, we are able to classify different flow regimes phenomenologically in a two-dimensional ($Fr; Re \cos \beta / Fr$)-plane and provide leading order expressions for the transitions between different regimes. The stabilizing and/or de-stabilizing effects of the imposed mean flow on buoyant exchange flows (zero imposed velocity) are described for a broad range of dimensionless parameters. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4903822]

I. INTRODUCTION

We present a computational study of high-Péclet-number miscible displacement flows in an inclined 2D plane channel with a heavier fluid displacing a lighter fluid downwards, i.e., density-unstable configuration. Iso-viscous Newtonian fluids in the Boussinesq limit are considered. This study continues our previous work on 2D channel displacement flows\(^1\)\(^2\) which have been focused at near-horizontal channels. Here our study covers the full range of channel inclinations ($\beta$) and extends from the exchange flow limit (zero imposed flow) into ranges with significant Reynolds number ($Re$). Thus, many of the flows are strongly inertial, driven by the imposed flow rate as well as by buoyancy (characterized by the densimetric Froude number, $Fr$). This leads to interesting physical phenomena, particularly in the early part of the flows, which we present later.

The principal focus of our study is on long 2D channels and characterizing the long-time behaviour of these flows in terms of the 3 principal dimensionless groups: $Re$, $\beta$, and $Fr$; see Sec. II A for definition of the dimensionless parameters. The motivation comes from industrial applications in oilfield cementing. In such applications, buoyancy and inertia are nearly always significant and displacement geometries have very long length-to-width aspect ratios ($\sim 10^3$–$10^5$). Although detailed physical understanding of different flow regimes is important and scientifically interesting, the end aim for industrial design of these processes is to estimate bulk flow features. Relevant here is the displacement efficiency (inversely related to the displacement front velocity, i.e., the velocity of the interpenetrating front of the heavy fluid into the light one) and the bulk axial diffusivity. The former is relevant in regimes that are primarily convective whereas the latter is important where the degree of transverse mixing is high.

\(^1\)Author to whom correspondence should be addressed. Electronic mail: kamran.alba@yahoo.com

1070-6631/2014/26(12)/122104/22/$30.00
c2014 AIP Publishing LLC
There is an extensive literature on miscible displacement flows in channels with little/no inertia, i.e., Hele-Shaw cells. In iso-dense displacements, dispersive regimes are found when the length to width aspect ratio ($e^{-1}$) exceeds the Péclet number ($Pe$). In the limit of $Pe \rightarrow \infty$ at finite aspect ratio, we approach immiscible displacements at infinite capillary number. In both lab experiments and applications, we commonly find finite ($Pe, e$) such that $Pe \gg e^{-1}$. Variable viscosity displacements were studied computationally by Goyal and Meiburg, using a linear stability analysis of developed two-dimensional Stokes flow displacements and showing the onset of spanwise instabilities. Three-dimensional computations are studied by Oliveira and Meiburg, revealing an inner splitting mechanism.

With density differences, downward displacement flow in a vertical Hele-Shaw cell was studied experimentally and theoretically by Lajeunesse and co-workers. Fluids were chosen to be miscible and Newtonian. Neglecting diffusive effects, the dependence of the flow stability on viscosity ratio and flow rate was investigated. In the limit of Stokes flow, Goyal et al. have carried out a computational study to compare with the results in Ref. 9. It was found that both the growth rate and the dominant wave-number depend only weakly on Péclet number and also that a moderately stable density stratification can stabilize a viscously unstable displacement (also see Ref. 12 for similar effects observed in iso-viscous pipe displacement flows with stable density stratification). More recently, Jiao and Maxworthy have studied fingering patterns and characteristics in viscosity and density unstable displacements. Exchange flows in Hele-Shaw geometries (i.e., zero imposed flow) are studied experimentally and computationally in Refs. 14 and 15; see also Refs. 16 and 17. Buoyancy effects in horizontal Hele-Shaw displacements have received more recent attention. Viscously unstable displacements were studied in three-dimensions by Talon et al. and John et al. The viscosity ratio leads to development of a finger that destabilizes along its length due to buoyancy effects. These may lead to pinch-off of the finger tip and to the formation of cavities. Very recently, Haudin et al. have studied viscosity-stable displacements, revealing interesting striped patterns resulting from buoyancy as the displacement front elongates.

Moving from the Hele-Shaw paradigm into regimes of significant inertia, Hallez et al. looked into exchange flows, i.e., flows with zero imposed velocity in channels and pipes using Direct Numerical Simulations (DNSs). In inertial regimes, they discovered that there are essential differences between 2D and 3D flows which arise from different dynamic vorticity fields. Sahu et al. have studied miscible displacement flows in 2D channels computationally. They considered the effects of varying the density ratio, the mean imposed velocity, and the channel inclination on the flow dynamics for moderate Reynolds numbers. Although numerous flow patterns were illustrated over the wide range of parameters chosen in Ref. 24, neither comprehensive picture of different phenomena emerges nor are quantitative predictions of the main flow features made. This motivates to study these flows further, even qualitatively in the case of a 2D channel, to provide better understanding of the underlying physical mechanisms that govern different flow regimes observed.

Two-dimensional channel flows present many advantages for study, both in terms of reduced complexity and computational speed. Although three-dimensional computations of miscible Newtonian fluid-fluid flows are feasible on large parallel machines, e.g., Refs. 19, 21, and 22, current computational speeds limit these studies to moderate aspect ratios and restricted ranges of dimensionless parameters. Applications of interest require the understanding of long time behaviour (meaning large aspect ratio) and eventually to include complex fluid rheologies, which increase significantly the number of dimensionless parameters.

A second advantage is that analytical and semi-analytical approximations can be made, allowing some degree of flow prediction, e.g., Refs. 1 and 9. These reduced models also lend themselves to stability analyses that may allow the prediction of flow regime transitions. For example, Sahu et al. considered the onset of convective instabilities in two-layer plane-channel flows. Analysis of the flow in the linear regime revealed the presence of both convective and absolute instabilities identifying the vertical gradient of viscosity perturbations as the main destabilizing factor. Through transient numerical simulations, the development of complex dynamics in the non-linear regime was demonstrated. The non-linear dynamics were characterized by roll-up phenomena and intense convective mixing which become pronounced with increasing flow rate and viscosity ratio, as well
as weak diffusion. In Ref. 26, we have extended the thin-film approach of Ref. 1 into the weakly inertial range, using a weighted-residual formulation. This allows the numerical study of convective instabilities, as well as linear stability analysis.

The pipe exchange flow (zero mean flow) has been extensively studied experimentally by Seon et al.27-31 The controlling parameter for flow stabilizing and/or de-stabilizing is the parameter $Re \cos \beta / Fr$. For $Re \cos \beta / Fr \gtrsim 50$, exchange flows become progressively inertial leading eventually to complete transverse mixing, whereas for $Re \cos \beta / Fr \lesssim 50$, density-stratified viscous regimes persist. Our own work has extended these studies into the displacement regime (non-zero imposed mean flow). In Ref. 32, we have classified flows as fully diffusive, inertial, and viscous, providing a qualitative description of each in the dimensionless planes of $Fr$ and $Re \cos \beta / Fr$. Some displacement fronts move instantaneously in the imposed direction, while others have buoyancy driven backflows, i.e., against the mean flow direction. Due to strong mixing near-vertical inclinations, instantaneous displacements can still exist at values of $\chi(=2Re \cos \beta / Fr^2)$ higher than predicted by the lubrication model derived in Ref. 33 for near-horizontal settings. For each type of flow, an approximation to the front velocity $\hat{V}_f$ and to the bulk axial diffusivity $\hat{D}_M$ was derived; see Ref. 32. The thin-film/lubrication model from Ref. 2 is effective in predicting $\hat{V}_f$ for viscous regimes. When the flow is fully diffusive, the front velocity is very close to the mean imposed velocity $\hat{V}_f \approx \hat{V}_0$. It was found that $\hat{D}_M$ exceeds the Taylor dispersion coefficient,34 $\hat{D}_T$, by up to an order of magnitude, over the full experimental range. Adding the flow rate to the exchange flow was found to have a stabilizing effect for $Re \cos \beta / Fr \lesssim 270$, extending the range studied in Ref. 35. Above this limit, the imposed flow was found to progressively destabilize the flow up to $Re \cos \beta / Fr \approx 500$. For larger values of $Re \cos \beta / Fr$, the imposed flow has little effect on the flow type, since the degree of mixing is already very high, i.e., the flow is fully diffusive.

In comparing with pipe flows, there are clear limitations. For example, in the turbulent pipe exchange flow simulations in Ref. 23, non-zero radial and azimuthal components of velocity were found that govern the mixing mechanisms. These (anisotropic) secondary motions are induced by a combined effect of the lateral walls and turbulent velocity fluctuations. Clearly lateral wall effects are lacking in any 2D study. The other important element (which is present in our study) is the existence of an asymmetric mean shear in the flow, which originates from the heavy-light configuration. In other words, the density difference in each layer modifies the mean pressure gradient driving the flow. Turbulent fluctuations produced along the streamwise direction can then be unevenly distributed along the other two directions. See also Refs. 36-38 for similar mechanisms in homogeneous shear flows. Therefore, at the outset of our study, we accept that there will be limitations in what may be inferred about fully inertial 3D displacements.

A plan of the paper is as follows. Below in Sec. II, we introduce the methodology and scope of the study. In presenting our results in Sec. III, we first discuss the main qualitative features of the displacement flows. Quantitative measurements of front velocities are given in Sec. III B. Instantaneous and non-instantaneous displacements are studied in Sec. III C. The phenomena happening at the displacement front are discussed in Sec. III D followed by macroscopic diffusion coefficient measurements in Sec. III E. Finally, we give overall classification of different regimes and a summary of the effects observed in Sec. IV.

II. DISPLACEMENT IN CHANNELS

Figure 1 shows the flow geometry and notation used in the current study. The fluids have the same viscosity but different densities. The displacing fluid is denser than the displaced fluid. The computations that we have carried out are fully inertial, solving the full 2D Navier-Stokes equations with the liquid species modeled via a scalar concentration, $C$,

$$[1 + \phi Ar] [u_r + u \cdot \nabla u] = -\nabla p + \frac{1}{Re} \nabla^2 u + \frac{\phi}{Fr^2} e_g, \quad (1)$$

$$\nabla \cdot u = 0, \quad (2)$$

$$C_t + u \cdot \nabla C = \frac{1}{Pe} \nabla^2 C. \quad (3)$$
Here, \( \mathbf{e}_g = (\cos \beta, -\sin \beta) \) and the function \( \phi(C) = 2C - 1 \) interpolates linearly between \(-1\) and \(1\) for \( C \in [0, 1] \). No slip conditions are satisfied at the walls, the heavy fluid enters fully developed (plane Poiseuille profile) at \( x = -L/4 \) and outflow conditions are applied at \( x = 3L/4 \).

**A. Parameters**

Five dimensionless groups appear in Eqs. (1)-(3). These are the Reynolds number \( \text{Re} \), densimetric Froude number \( \text{Fr} \), channel inclination \( \beta \), Atwood number \( \text{At} \), and Péclet number \( \text{Pe} \). We adopt the convention of denoting dimensional quantities with a \(^*\) symbol (e.g., the channel width is \( \hat{D} \)) and dimensionless quantities without. We denote the density of the displacing fluid by \( \hat{\rho}_H \) and that of the displaced fluid by \( \hat{\rho}_L \). The Atwood number is defined as \( \text{At} = (\hat{\rho}_H - \hat{\rho}_L)/(\hat{\rho}_H + \hat{\rho}_L) \), representing a dimensionless density difference. For this study, \( \text{At} > 0 \) since \( \hat{\rho}_H > \hat{\rho}_L \) (heavy fluid displacing the light fluid in the downwards direction). Our simulations are all performed for small \( \text{At} = 0.001, 0.0035, 0.01 \), the significance of which is that a Boussinesq approximation is valid. Briefly, this means that density differences can significantly affect buoyancy forces, captured by the densimetric Froude number, \( \text{Fr} = \hat{V}_0/\sqrt{\text{At} \hat{g} \hat{D}} \), but do not significantly affect the acceleration of the individual fluids. Here, \( \hat{g} \) is the gravitational acceleration and \( \hat{V}_0 \) is the mean imposed velocity. The Reynolds number is defined as \( \text{Re} = \hat{V}_0 \hat{D}/\hat{\nu} \), where \( \hat{\nu} \) is defined using the mean density \( \hat{\rho} = (\hat{\rho}_L + \hat{\rho}_H)/2 \) and the common viscosity of the two fluids, \( \hat{\mu} \). Finally, the Péclet number \( \text{Pe} = \hat{V}_0 \hat{D}/\hat{D}_m \) (here, \( \hat{D}_m \) is the molecular diffusivity), and typically we have \( \text{Pe} \gg 1 \). This suggests that on the timescale of interest, molecular diffusivity does not play a major role in the flows studied. In summary for the high-Péclet-number Boussinesq regime considered, the key governing dimensionless parameters of the problem reduce to \( \text{Re}, \text{Fr}, \) and \( \beta \).

Approximately 180 simulations have been carried out covering the parameter ranges indicated in Table I. Our dimensionless parameters are achieved dimensionally by varying mean velocity \( \hat{V}_0 \) (mostly in the range 0–27 (mm/s)) and fluid densities for a fixed viscosity \( \hat{\mu} = 10^{-3} \) (Pa s) in a channel of fixed width \( \hat{D} = 19 \) (mm). This mimics the scope and procedure of our pipe flow experiments in Ref. 32. Note that typically \( \text{Re} \leq 500 \) for our simulations. At larger \( \text{Re} \), we would expect to enter a fully mixed turbulent regime, for which we have not explored the performance of...
our code. It is worth noting that in a few cases, the flow rate was increased up to $Re = 800$ in order to cover a wider range of dimensionless conditions and better understand the flow dynamics. In presenting our results in the upcoming sections, data from the simulations from our previous study on displacement flows in near-horizontal channels have also been included for completeness and to help understand the effect of the inclination angle.

### B. Computational code

Equations (1)-(3) are discretised using a mixed finite element, finite volume method. The Navier-Stokes equations are solved using Galerkin finite element method. The divergence-free condition is enforced by an augmented Lagrangian technique. A fixed time step is used for the Navier-Stokes equations, advancing from time step $N$ to $N + 1$. The convective velocity is approximated at time step $N$ while the linear spatial derivatives of the velocity are approximated implicitly at time step $N + 1$. The pressure is approximated at time step $N + 1$ (semi-implicit method regarding the nonlinear terms). More details of the computational methodology and the governing system of equations are given in Ref. 2. To check the accuracy of the code, various simple test problems have been implemented. These include quantitative comparisons with the computational work of Sahu et al., for displacement flows and with Hallez and Magnaudet for exchange flows in 2D channels. The same code has been used extensively in Ref. 40 formiscible core-annular Newtonian flows of differing viscosities in which pearl and mushroom instabilities develop. The results therein have been benchmarked against the recent experiments of Refs. 41 and 42.

As explained earlier, we are mostly interested in predicting global features of the displacement flow. For the meshes in most of the computations, we have used 28 cells across the channel of height 19 (mm), refined slightly towards the walls, and 400 cells along the $L = 2$ (m) length of the channel. The initial interface between the two fluids in most cases was placed at a distance $L/4$ from the left side of the computational domain, where $L = L/D \approx 10^2$ is the dimensionless channel length (see also Fig. 1). The most obvious global feature of the flow is the average value of the concentration, defined as

$$\bar{C}(t) = \frac{1}{L} \int_{x=0}^{1} \int_{y=L/4}^{3L/4} C(x, y, t) \, dx \, dy.$$  

Figure 2(a) shows the dependency of $(1 - \bar{C}(t))/(1 - \bar{C}(0))$ and the position of the displacing front, $x_{\text{front}}$, on different mesh sizes, respectively, for parameters $\beta = 40^\circ$, $Re = 50$, and $Fr = 0.1$. Naturally at $t = 0$, the quantity $(1 - \bar{C}(t))/(1 - \bar{C}(0))$ approaches 1 and it decreases with time as more displacing fluid (with $C = 1$) is introduced into the computational domain. In Fig. 2(a), it is shown that this large scale feature of the flow does not depend on the mesh size significantly. Note that the slope of the lines shown in Fig. 2(a) changes at the breakthrough time, when the displacing front reaches the end of the computational domain (see also Ref. 24). After the breakthrough time, the only mechanism through which the displaced layer is removed from the channel is by the action of the displacing fluid on the displaced fluid. In viscous regime flows, this is by dragging the residual layers along the wall, through a combination of viscous and pressure gradient effects. In the case of unstable flows, this is augmented by mixing effects.

At $t = 0$, the two fluids are separated by an imaginary gate valve located at $x = 0$. As time progresses, due to the imposed mean velocity, the heavy layer penetrates through the light layer...
FIG. 2. (a) Evolution of \( (1 - \bar{C}(t))/(1 - \bar{C}(0)) \) with time for different mesh sizes for \( \beta = 40^\circ \), \( Re = 50 \), and \( Fr = 0.1 \). Note that \( \bar{C}(t) \) is the average concentration over the whole computational domain computed at time \( t \). (b) The evolution of the displacing front position, \( x_{front} \), with time for different mesh sizes for the same parameters as part (a). The broken vertical line shows the transition between different phases of the displacement.

and displaces it. The streamwise distance of the tip of the advancing displacing front from the gate valve increases with time and is denoted by \( x_{front} \) (initially, \( x_{front} = 0 \)). Figure 2(b) shows \( x_{front} \) for the same range of mesh sizes as in Fig. 2(a). First of all we can see that for this particular simulation, the rate of increase of the front position experiences a different behavior at \( t \approx 1 \) (indicated by red vertical dashed-line). Initially, mixing is controlled by inertia due to the very strong buoyancy forces. This is the reason why the front accelerates faster at the beginning. Over time, viscous forces come into balance with inertia and buoyancy and the movement of the front becomes more steady. See also Refs. 30 and 43 for transient behavior of similar gravity currents. Second, we can see that the position of the front does vary slightly with the mesh size. This is because the mixing flows are better resolved in fine mesh sizes than the coarse ones. The relative error in final front velocity value is estimated to be of order 6%.

In the results that follow, we have mostly used mesh size \( 28 \times 400 \), in order to generate a broad data set of simulations covering the key parameters. This resolution is also consistent with our previous study (see Ref. 2), enabling direct comparison but for a fuller range of channel inclinations. To directly study particular features of interest (e.g., Rayleigh-Taylor-like instabilities at low \( \beta \)), we have used a finer mesh to ensure accuracy. In general, we have found that although the fine mesh simulations for these flows were different from the coarse mesh ones, from the quantitative point of view at fine scales, the large scale and qualitative features remained the same. The parameter ranges studied are selected to be similar to our experimental study in Ref. 32. For these miscible displacements, we expect to have Péclet numbers in the range \( Pe \gtrsim 10^6 \). In our previous work, we have explored the relative effects of numerical and physical diffusion, finding that for these mesh sizes, we are unable to physically represent Péclet numbers greater than \( \approx 10^5 \). Consequently, we have set \( Pe = \infty \) for our study. Observed diffusion is numerical, but note that the secondary flows and instabilities, that are largely responsible for dispersion, are physical.

As a benchmark example, in vertical channels and for small enough imposed velocity, we expect to observe flows that are phenomenologically similar to the Rayleigh-Taylor type instability; see, e.g., Ref. 44. Figure 3(a) shows concentration maps of a displacement flow in a strictly vertical channel (\( \beta = 0^\circ \)) for \( Re = 50 \) and \( Fr = 0.19 \). The figure clearly shows the formation of a planar Rayleigh-Taylor mushroom structure as time evolves. Due to both mixing and the imposed flow, the mushroom-like structure is not sustained over longer times and symmetry is lost after a short period (here \( \approx 10 \) s). Figure 3(b) shows the corresponding velocity fields to the concentration maps depicted in part (a). Note the upstream velocity vectors close to the startup interface where the mushroom initially forms.

The mushroom patterns shown in Figs. 3(a) and 3(b) were obtained for a strictly vertical channel. It is interesting to see if the same effect is observed at small inclinations away from vertical. Figure 3(c) shows concentration maps of simulations obtained for the same parameters as in Fig. 3(a) except for \( \beta = 10^\circ \). We see that due to the effects of buoyancy, the Rayleigh-Taylor
FIG. 3. (a) Concentration maps of the numerical simulation at times \( t = 0, 0.9, \ldots, 3.6 \) run for \( \beta = 0^\circ, Re = 50 \), and \( Fr = 0.19 \) showing the formation of Rayleigh-Taylor instability. The last image at the bottom of the figure is the colourbar of the concentration values. The size of the domain shown is \( 1 \times 13.68 \). (b) Velocity vector field with time for the same simulations shown in part (a). The dashed lines show the imaginary position of the gate valve in both figures. Figures (c) and (d) are run for the same parameters as to (a) and (b) except \( \beta = 10^\circ \). The mushroom pattern does not form anymore, i.e., symmetry is broken in the initial onset of flow, even though \( \beta \) is relatively small. Figure 3(d) shows the corresponding velocity vectors, with an upstream component to the velocity when close to the gate valve. The simulations shown in Fig. 3 were obtained for a mesh size \( 56 \times 1000 \). The same qualitative patterns were observed for the usual mesh size of \( 28 \times 400 \).

III. RESULTS

The main computational results are presented in this section. A broad phenomenological description of the main regimes observed in our displacement flow simulations is first given in Secs. III A and III B. These flow regimes are quantified through numerous tools, e.g., spatio-temporal plots, velocity fields, time dependent interface evolution profiles, etc. This leads to a delineation of the observed flows into a number of regimes, the boundaries of which we are able to identify in terms of the main dimensionless groups: \((Re, Fr, \beta)\) (see Secs. III C and III D). In the case of transversely mixed flows, we analyze the effective axial diffusivity (or dispersivity), from measured concentration profiles in Sec. III E.

A. Displacement flows: Main qualitative features

We first address the most basic question: namely, is the global qualitative behavior observed for pipe displacement flows, as shown in Fig. 4(a), significantly affected by introducing the new geometry? Fig. 4(b) shows concentration maps from typical results from our simulations, primarily
FIG. 4. Change in iso-viscous displacement flows with $\beta$ for (a) experiments in circular pipe $^{32}$ for $Re \approx 726$ and $Fr \approx 1.48$ and (b) simulations in 2D channel (current study) for $Re = 300$ and $Fr = 0.61$. The field of view in (a) is $1 \times 70$ located at $x = 91$ downstream of the gate valve. The field of view in (b) is $1 \times 75$ with the white dashed lines indicating the position of the imaginary gate valve. The concentration maps are taken at $t \approx 160$ and 20.6 in (a) and (b), respectively. The color bar at the top left of the figure shows the corresponding concentration value, $C$, with 0 referring to the pure displaced fluid and 1 to the pure displacing fluid.

illustrating the effect of inclination angle $\beta$ on the flow pattern. The parameters range in Figs. 4(a) and 4(b) are chosen to be fairly close for comparison. Although we see a transition from fully transversely mixed flows ($\beta = 0^\circ$) to segregated viscous flows ($\beta = 90^\circ$), there is also the suggestion that the 2D channel flow displacement seems to destabilize more than the pipe flows. The degree of mixing even in near-horizontal ducts seems to be higher in 2D channels than in analogous pipe flows. Note that the viscous flow behavior similar to that shown in Fig. 4(b) for $\beta = 90^\circ$ is also observed for $\beta$ down to $85^\circ$ in our simulations.

Another key difference between 2D channel and pipe flow displacements is that the slumping pattern is more pronounced in the latter. By slumping, we mean that the heavy layer slouches underneath the light layer and travels downward closer to the lower channel wall. A slump-type flow is similar to a two-layer flow. In fact, we generally observe a two-layer flow in our experiments whereas in 2D channel flow simulations, the two-layer structure is not observed as frequently (note that even for the case of fully transversely mixed flows in pipe, the initial phase of the displacement is slumping, with instabilities growing rapidly over time at the interface between the two fluids). In fact, the 2D channel flows computed are more similar to a three-layer structure than to a slumping two-layer flow. This can be purely due to geometrical effects. Possibly in pipe flows, as the displacing liquid moves downstream, it squeezes the displaced layer around the pipe. This in turn helps the displaced layer to direct the displacing fluid down towards the lower pipe walls (strengthening the slumping effect). However, in 2D channel geometry, there is no such geometrical limitation that generates azimuthal motions, i.e., the flow is 2D and thus the displacing liquid cannot squeeze the displaced layer towards the upper wall in the same way as in pipe (see also Ref. 2 for similar effects in near-horizontal ducts).

Hartel et al. $^{45}$ studied the flow at a gravity-current head in a 3D channel through analysis and direct numerical simulations. It was shown that the flow disturbance could extend across the whole channel span (cross-stream direction). Perhaps in a 3D channel, instabilities observed in Fig. 4 would develop in the cross-stream directions as well as in the streamwise direction. It is also worth noting that similar to the pipe exchange flows of Ref. 29, the degree of mixing and disorder in the system increase as we move towards the vertical ($\beta \to 0^\circ$). The flow pattern for more horizontal inclinations remains qualitatively similar to viscous flows of two separated layers; see $\beta = 90^\circ$ in Fig. 4(b). Note that in the case of viscous flows detecting an interface between the fluids is much easier than the case where mixing and instabilities are strong. By interface, we mean a point or points at a streamwise location where the concentration gradient transversely is high and the two fluids meet and interact.

Similar to the previous study for pipe displacement, $^{32}$ we now try to investigate the physical features of the flows shown in Fig. 4(b) in more details. Figures 5(a) and 5(b) show the spatiotemporal diagrams constructed from depth-averaged concentration fields $\bar{C}_y$ for the same simulations as...
FIG. 5. Spatiotemporal diagrams of the depth-averaged concentration field $\bar{C}_y$ obtained for the same simulations as shown in Fig. 4(b) for (a) $\beta = 0^\circ$ and (b) $\beta = 90^\circ$. The dashed line in part (b) indicates the position of the displacing front and its slope is $-1/V_f$. In this case, the spatiotemporal diagram gives $V_f = 1.44$ and the method to be explained later in Sec. III B predicts $V_f = 1.43$.

shown in Fig. 4(b) for $\beta = 0^\circ$ and $90^\circ$. Note that

$$C_y(x,t) = \int_0^1 C(x,y,t) \, dy.$$ 

The depth-averaged concentration $C_y(x,t)$ does not give any information whether or not the flow is symmetric in the transverse direction. However, it provides us with some very useful information about how much displacing and/or displaced fluids exist in a given streamwise location, $x$, at time $t$. The closer $C_y(x,t)$ is to 1, the more displacing fluid exist at location $x$ and time $t$. If, on the other hand, $C_y$ is closer to 0, it means that most of the 2D channel width is occupied by the displaced fluid. As will be seen later in this section, $C_y$ is also used in spatiotemporal diagrams to estimate the flow stability. Another importance of $C_y$ is in giving the location of the advancing displacing front, $x_{\text{front}}$, and finally its speed, $V_f$.

As can be seen in Figs. 5(a) and 5(b), there are less waves appearing in the spatiotemporal diagram of the concentration field as we move toward near-horizontal inclinations. The existence of the waves in the concentration field map can be related to the stability of the flow (see also Fig. 4(b)). If the transition from purely displacing fluid region, $C = 1$, to the purely displaced fluid one, $C = 0$, in the spatiotemporal diagram happens smoothly (Fig. 5(b)), it means that the interface between the two fluids is also smooth with no waves and instability appearing between the two fluids. In principle, we could extract statistical data on the wavelengths and growth rates of the instabilities from this data, but this is not the aim of the present study. In contrast to the experimental spatiotemporal diagrams shown previously in Ref. 32, even when the degree of mixing is high it is not very difficult to distinguish the leading displacement front from the numerical simulations. Of course, when the fluids become more separated, the location of the advancing displacement front can be recognized clearly (Fig. 5(b)). The inverse slope of the line bounding the blue region in the spatiotemporal diagram is minus the front velocity (see the dashed line in Fig. 5(b)). This boundary can be either obtained through standard image processing methods or through the concentration evolution profiles discussed later in Sec. III B.

B. Front velocity measurement and characteristics

In a long channel, the ratio of the mean flow speed to the leading front velocity, $V_0/V_f$, indicates the proportion of the channel displaced in an experiment or a simulation, i.e., the displacement efficiency. Consequently, it becomes important to measure $V_f$ in a consistent and repeatable way, regardless of the degree of mixing. In our previous experimental study, 32 we introduced a robust method to find the displacing front velocity using the profiles of the depth-averaged concentration values. Following the same method as explained in Ref. 32, we measure the front velocity
values for our numerical simulations. In contrast to the experiments, where there was noise in the measured concentrations close to the lower wall of the pipe meant setting a relatively high threshold ($\hat{C}_y = 0.01$) for detection, we do not have such problem in the numerical solutions. However, to avoid numerically diffuse concentrations, we estimate the speed of the displacement front by the velocity of the concentration level set $\hat{C}_y = 0.01$. It is also verified that the choice of concentration threshold used to compute $V_f$ does not influence the front velocity significantly.

Similar to the experiments, initially the front velocity accelerates as the flow initiates due to large buoyancy forces across the initial interface. As time passes, viscous and inertial forces start to balance the buoyant forces. In those cases that instability and mixing cause slow oscillation in the profile of front velocity with time, or if the front velocity slightly decreases or increases with time, an average value is adopted taken over long times so that our front velocity values are representative of well-developed flows. We later use the measured long-time front velocity to understand the large scale features of the flow and the dominant dynamic trends.

Following the same method for measuring the front velocity as in Ref. 32, we now present a dimensionless analysis of the results, focusing on the normalized front velocity $V_f$. In our previous study of near-horizontal displacements, we were able to classify all flows in the $(Fr, Re \cos \beta/Fr)$-plane, and therefore start with this description. The competition between inertia from the mean flow and buoyancy is captured in $Fr = \hat{V}_0/\sqrt{\hat{At} \hat{g} \hat{D}}$. On the other hand, the competition between buoyancy-driven advection and transverse mixing is captured by $Re \cos \beta/Fr$ defined as

\[
\frac{Re \cos \beta}{Fr} = \frac{\hat{V}_0 \hat{D}}{\hat{\nu}} \cos \beta \sqrt{\frac{\hat{At} \hat{g} \hat{D}}{\hat{V}_0}} = \frac{\cos \beta}{\sqrt{2}} \left[ (\hat{\rho}_H - \hat{\rho}_L)(\hat{\rho}_H - \hat{\rho}_L) 2 \hat{\mu}^2 \right]^{1/2}.
\]

The parameter in the square brackets above is the Archimedes number. An interpretation of this expression is as the relative strengths of buoyancy stresses, $(\hat{\rho}_H - \hat{\rho}_L) \hat{g} \hat{D}$, and viscous stresses, $\hat{\mu} \hat{V}_f/\hat{D}$. Figure 6(a) shows the normalized front velocity $V_f$, plotted against $Fr$ and $Re \cos \beta/Fr$, for all simulations run in this study (Figure 6(b) will be explained in Sec. III C). The scale is adjusted up to $V_f = 2$. Approximately 100 simulation data points belonging to the near-horizontal displacements studied by Taghavi et al. are also included in the figure for comparison marked by solid black circles around data points, mostly located in the area $Re \cos \beta/Fr < 100$. It can be seen that the present study covers a much wider parameter range, primarily due to variations in $\beta$. Concentration maps added to Fig. 6(a) show a characteristic exchange dominated flow with $V_f > 2$ and a flow with $V_f < 2$.

**FIG. 6.** (a) Presentation of our results for the full range of simulations: normalized front velocity $V_f$, plotted against $Fr$ and $Re \cos \beta/Fr$. The simulations belonging to the near-horizontal displacements study are also included for comparison marked by solid black circles around data points (mostly in the area $Re \cos \beta/Fr < 100$). Concentration maps show a characteristic exchange dominated flow with $V_f > 2$ ($\beta = 80^\circ$, $Re = 50$, and $Fr = 0.19$ at $t = 5.5$) and a flow with $V_f < 2$ ($\beta = 20^\circ$, $Re = 500$, and $Fr = 1.92$ at $t = 19.25$). (b) Classification of our results for the full range of simulations plotted in the plane of $Fr$ and $Re \cos \beta/Fr$. The instantaneous displacement flows are marked by blue squares and non-instantaneous ones by red circles. The heavy line represents the prediction of the lubrication model of Ref. 1 for the stationary interface ($\mathcal{H}_L = 2Re \cos \beta/Fr^2 = 69.94$). The arrows indicate that the non-instantaneous flows zone shrinks at higher $Re \cos \beta/Fr$ values due to a more efficient mixing. Concentration maps show characteristic instantaneous ($\beta = 10^\circ$, $Re = 500$, and $Fr = 1.92$ at $t = 19.25$) and non-instantaneous ($\beta = 60^\circ$, $Re = 50$, and $Fr = 0.19$ at $t = 5.5$) flows. White solid lines in all concentration maps indicate the position of the gate valve and size of the domain shown is $1 \times 4$. 

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 129.7.158.43 On: Thu, 26 Mar 2015 22:26:42
The parameter \( \text{Re} \cos \beta / Fr \) is independent of \( \hat{V}_0 \), so that as \( Fr \to 0 \), we approach the exchange flow limit. For small \( Fr \), we observe a good number of flows for which \( V_f > 2 \). We refer to these as *exchange flow dominated* similar to the pipe flow experiments and note that since we have scaled with the mean flow velocity, large \( V_f > 2 \) strongly suggests that some part of the velocity field is moving backwards against the mean flow, driven by buoyancy. We have checked in our numerical simulations that for the flows with \( V_f > 2 \), there always exists a layer of the lighter displaced fluid moving backwards up the position of the gate valve and against the mean flow direction. More discussion on the back-flow will be given in Sec. III C. As will be seen later, back-flow can also happen for \( 1 < V_f < 2 \). Figures 9(a) and 10(a) show some examples of simulations with back-flow.

As mentioned earlier, the 2D channel displacement flows show more instability and mixing compared to those in the pipe geometry. In the case of the pipe displacement when the mixing is efficient, the front velocity value is very close to that of the mean flow, i.e., \( V_f \approx \hat{V}_0 \). However, this seems to be quite different for 2D channel flow. In fact, when the fluids mix significantly, the displacing front still advances close to the channel center; see Fig. 13, for example. One then expects to have \( V_f \approx 1.5 \hat{V}_0 \) (by analogy with the Poiseuille flow). This effect is also evident in Fig. 6(a). For highly unstable flows with strong mixing that often appear for large \( \text{Re} \cos \beta / Fr \) and/or large \( Fr \), the normalized front velocity, \( V_f \), is still larger than 1 (and close to 1.5). In the case of pipe flow, for mixing and unstable flows, \( V_f \approx 1 \); see Ref. 32. As a side note, if we were to run the simulations over longer channel length and at larger times, the mixing could have been stronger and more complete transversely. In that case, the front velocity value might approach the mean flow speed \( V_f \approx \hat{V}_0 \), similar to the pipe flow experiments.\(^{32}\) Finally, note that a general trend cannot be drawn in Fig. 6(a) for the variation of \( V_f \) in the plane of \( Fr-\text{Re} \cos \beta / Fr \). It seems as though the front velocity, \( V_f \), can either increase or decrease with \( Fr \), depending on the range of \( \text{Re} \cos \beta / Fr \).

C. Instantaneous and non-instantaneous displacements

An important large scale feature of displacement flows is whether they are *instantaneous* or not. By instantaneous displacement, we mean that the displaced fluid does not travel upstream of the gate valve once the simulation/experiment starts. If the displacement is not instantaneous, this implies a back-flow of the displaced fluid layer against the mean flow. From the industrial point of view, this characteristic is very important if it is sustained, meaning a residue of the displaced fluid. Ideally, we want to always displace instantaneously and avoid back flow. Taghavi et al.\(^{33}\) studied flows with back flow in detail, for near-horizontal channels. It was found that the *trailing front* can show different behaviors as the imposed velocity was varied. The trailing front is that which moves close to the upper wall of the channel, moving slower than the leading front. The possible movements of the trailing front include instantaneous displacements, stationary layers, temporary, and sustained back flows. By slowly increasing the mean flow from 0, different patterns can arise. For very low values of imposed velocity, the trailing front keeps moving upstream of the gate valve due to the strong buoyancy force relative to that of the imposed flow. If the imposed flow is further increased, the trailing front moves upstream of the gate valve but then stops at some location \( x \). The trailing front may remain stationary, at a critical \( \hat{V}_0 \), but for slightly larger \( \hat{V}_0 \), the trailing front returns back downstream after some time \( t \). Finally, if the mean flow is sufficiently high, the trailing front is displaced instantly by not flowing backwards, upstream of the gate valve at all. Channel displacement flows at higher inclinations have not been studied from the perspective of the trailing front so far. In the current simulations, we confirmed that stationary layers may also exist at larger inclinations where the degree of mixing and instability are higher. However, the results are not presented here, to avoid repetition (see Ref. 33 for similar effects).

We now classify our displacement flows in terms of being instantaneous or non-instantaneous, over the range of parameters computed. Figure 6(b) shows such a classification in the dimensionless plane of \( Fr \) and \( \text{Re} \cos \beta / Fr \). The blue squares’ data in this figure indicate instantaneous displacement and all others are non-instantaneous flows. The concentration maps added show characteristic flows. The prediction from the lubrication model of Ref. 1 is also added as a line to the graph. The pattern shown in Fig. 6(b) is qualitatively similar to that of the experimental study presented in Ref. 32. As \( Fr \) increases, the flows transition from non-instantaneous to instantaneous
Displacements. The required Froude number for this transition decreases with \( Re \cos \beta / Fr \) at least when away from near-horizontal displacements (small \( Re \cos \beta / Fr \)). This might first seem counter-intuitive because at inclination angles closer to the vertical, one expects an enhanced counter-current flow due to the buoyancy. The required mean flow velocity (or correspondingly \( Fr \)) then required to overcome the back flow also is expected to be higher. However, we observe that the minimum Froude number required to transition to instantaneous flows actually decreases with \( Re \cos \beta / Fr \). The reason is that although the inclination angle is getting closer to the vertical, increased instability and transverse mixing in the system do not allow the displaced layer to travel upwards the gate valve. In fact, the instability induces a strong transverse mixing and suppresses the back flow and counter current motion. This is somehow reminiscent to the famous Boycott effect. The arrows added to Fig. 6(b) help show this phenomenon better. The transition from the sustained back flow displacement flows to the instantaneous displacement flows happens over a very narrow range of the imposed mean flow and the Froude number. In order to exactly capture the transition range, including all the stationary interface and temporary back flows, a very detailed computational programme is required.

In the picture given in Fig. 6(b), most of the data marked as non-instantaneous flows belong to the category of the sustained back flows. For industrial applications and as far as long-time behaviour of the flows is concerned, the temporary back flows can be considered as instantaneous displacement flows since the displaced fluid will finally be removed from the section above the position of the initial interface between the two fluids. Also note that in the limit of the exchange flows (\( Fr \rightarrow 0 \)), there is always a back flow meaning that the picture in Fig. 6(b) can easily be extended to \( Fr = 0 \). Studying the exchange flows is not in the scope of the current paper. However, throughout this study, some remarks on the exchange flow are given where appropriate.

D. Displacing front phenomena

In the case of iso-viscous Newtonian displacement flow in pipe,\(^{32}\) the leading displacement front remained connected to the bulk of the displacing fluid. Even in the case of mixing and instability, although waves appeared at the interface between the fluids, there was no cutting of the stream between the displacement front and the rest of the displacing liquid. In the case of channel displacements, we have observed a number of different phenomena at different inclination angles and/or flow rates. In this section, we describe these phenomena and specify their location in the \((Fr, Re \cos \beta / Fr)\)-plane.

1. Front detachment

One of the most common behaviors observed throughout the numerical simulation results was that the displacement front seemed to be cut from the rest of the displacing layer. Figure 7 shows an example of this for \( \beta = 20^\circ \), \( Re = 500 \), and \( Fr = 1.92 \). Figure 7(a) shows the concentration maps of the concentration field at different times. As shown in the figure, the front initially is attached to the larger body of displacing fluid, but over time as instabilities start to form, it is cut off from the rest of the displacing layer. Evidently, as well as the instability mechanism to cut the displacing fluid stream, the detached front appears to move faster than the bulk, being analogous to a falling droplet.

Figure 7(b) shows the velocity vectors corresponding to the same concentration map as in part (a). Although there is an apparent change in concentration close to the frontal region, evident from the concentration maps, the velocity field seems to vary smoothly in that region. Also note the unstable nature of the velocity field in this case. Figure 7(b) shows that due to the instabilities, the depthwise component of the velocity changes sign over time at a location. Although the front is separated from the bulk of the displacing fluid initially, it mixes with the displaced fluid over longer times and finally diminishes (Fig. 7(a)). In order to better understand the formation and diminishing processes, the concentration values are averaged across the channel at each location and are plotted in Fig. 7(c). These profiles correspond to the same instants of time as in parts (a) and (b). It can be seen that there is an obvious change in concentration close to the frontal region as the displacing front is cut by the instabilities. Mean concentration profiles in 3D pipe flows are also known to display bulges at the front with a mean concentration of displacing fluid decreasing.
FIG. 7. (a) Concentration maps of the numerical simulation at times $t = [6.8, 17.9, \ldots, 40]$ run for $\beta = 20^\circ$, $Re = 500$, and $Fr = 1.92$ showing the detachment of the front from the displacing fluid layer. The last image at the bottom of the figure is the colourbar of the concentration values. The size of the domain shown is $1 \times 63.5$ starting from gate valve position $x = 0$. (b) Vectors of velocity field with time for the same simulations shown in part (a). (c) Evolution of the depth-averaged concentration field, $\bar{C}_y$, with time and streamwise location for the same simulation as in parts (a) and (b). The arrows point the position of the detached front that disappears over time. (d) Spatiotemporal diagram of the averaged concentration values, $\bar{C}_y$, for the same simulation clearly showing the detachment of the front at different times and streamwise location.

Before increasing again.\textsuperscript{21} Finally, Fig. 7(d) shows the spatiotemporal diagram of the averaged concentration values, $\bar{C}_y$, for the same simulation showing the evolution of the detached front with time and streamwise location. In accordance to Fig. 7(c), the front is first separated from the bulk of the displacing fluid but mixes with the displaced fluid and diminishes over time.

The type of instabilities observed in Fig. 7(a) is quite common in channel displacement flows leading us to take a closer look at these instabilities and extract more physical information from the available data. Figure 8 shows a representative case from the same numerical simulation as Fig. 7, at time $t = 40$. Figure 8(a) is basically the last concentration map in Fig. 7(a) plotted on a larger scale for convenience. Figure 8(b) shows the contour of the speed $V = \sqrt{V_x^2 + V_y^2}$ (here, $V_x$ and $V_y$ are the dimensional streamwise and depthwise velocity components, respectively). We can see that although the velocity is far from a Poiseuille profile, the high speed regions remain towards the center of the channel, with unstable oscillations mimicking those in the concentration.

Figure 8(c) shows the vorticity contours, $\omega$, for the same concentration map as in part (a),

$$\omega = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}.$$  

We see that positive vorticity is mostly located on the upper half of the channel whereas the negative vorticity is in the lower half, as would be expected for a channel flow. However, on close inspection, we see that there are also positive values of $\omega$ close to the lower wall and negative values close to the top wall, which is not expected. This indicates the existence of tiny rotational regions of current causing local back flow and mixing. The main cause of these local back flows must be the instabilities close to the interface and within the fluid layers. To see this interesting effect better, we have

\[\text{ attachment not provided in text }\]
plotted in Fig. 8(d) the velocity vectors focusing on this back-flow area highlighted in Fig. 8(c). The figure clearly shows the velocity reversal close to the lower wall of the channel causing positive vorticity. Note that vorticity is generated close to front and decays a bit behind the front. Just an additional comment here is that the type of instabilities and the related mechanisms are very similar when either fine or coarse meshes are used.

Another interesting phenomenon observed in a few simulations under the category of flows with front detachment was the formation of a second displacing front from within the displacing fluid region. Figure 9 shows an example of this for $\beta = 80^\circ$, $Re = 300$, and $Fr = 0.61$. Figure 9(a) shows the concentration maps of the simulations at successive times. At time $t \approx 18.6$, an extra tip region starts to form at the displacing front. Figure 9(b) shows the interface evolution profiles for the same concentration maps as in part (a). The figure clearly shows that a jump in the averaged concentration value starts to form close to the front. Again the results presented in Fig. 9 are for a finer mesh of size $56 \times 1000$. However, qualitatively similar results were obtained for the coarser mesh size of $28 \times 400$. 
2. Oscillating front speed

As mentioned earlier, channel flow displacements in the current study were run over a range of imposed velocities $Re \in [50 - 500]$. For small imposed velocities, we inevitably recover results close to the exchange flow of two fluids; see Refs. 27–31 and Refs. 21–23 for exchange flow experiments and numerical simulations, respectively. Hallez et al. looked into exchange flows in both channel and pipe using direct numerical simulations. They discovered that there are essential differences in 2D and 3D flows which arise from different dynamic vorticity fields. From this perspective, strong and coherent vortices found in channel flows enable the vortices to cut the layers of pure fluid that feeds the fronts in a periodic manner. In contrast, the vortical motions in 3D are such that they promote segregation effect and thus avoid completely cutting the pure fluid layers; see Ref. 21. In our numerical simulation for 2D channel displacement flows, we also observed this type of oscillatory behavior in the leading front velocity. Figure 10 shows a representative case obtained for $\beta = 20^\circ$, $Re = 50$, and $Fr = 0.06$. Figure 10(a) shows the concentration maps of the simulations where the mixing is fairly strong. Due to the low imposed velocity in this case, it is very likely that we are in exchange dominated regime. The flow instability and back flow are common features of such flows, at least at higher inclination angles. Figure 10(b) shows the corresponding spatiotemporal diagram of the same simulations. The figure shows unevenness in the boundaries of the two displacement fronts, due to oscillation. Note that the image contrast is adjusted here to show the slight variation and unevenness in concentration values better. The uneven oscillations in the depth-averaged concentration shown in Fig. 10(b) reflect the oscillations present in the front velocity due to the vortical structures.
Figures 10(c) and 10(d) show the leading and trailing front velocities, which exhibit clear oscillations. The peaks observed for $t \approx 1$ in Figs. 10(c) and 10(d) relate to the initial acceleration of the flow. The flow has to accelerate from 0 initially at $t = 0$ to a finite value $V_f$ for $t > 0$. Inevitably, there is going to be a growth in $V_f$ during this development period which is shown as two peaks in Figs. 10(c) and 10(d). As shown in Fig. 10(c), at longer times, the value of the leading front velocity fluctuates around an almost-constant value ($V_f \approx 0.09$). The oscillations observed repetitively cut the pure fluid layers feeding the fronts. This cutoff in the feeding layers can accelerate and/or decelerate the front (see Ref. 21 for similar effects in exchange flows). Figure 10(d) shows a similar phenomenon at the trailing front, where also the pure fluid layers are cut. Note that here we have a small imposed velocity that breaks the symmetry of the two fronts. As before, the simulations in Fig. 10 were obtained for a mesh size $56 \times 1000$, but with a qualitatively similar oscillatory pattern in front velocity observed for coarser mesh size of $28 \times 400$.

3. **Dispersive horn formation**

In the case of near-horizontal channels, another interesting displacement front pattern was observed. Figure 11 shows the existence of a dispersive horn-like region at the front. Figure 11 shows concentration maps of a typical simulation carried out for $\beta = 90^\circ$, $Re = 200$, and $Fr = 0.77$. At time $t \approx 19.8$, the horn starts to form and extends progressively. Always about the interface, we have secondary flows that tend to disperse fluid that diffuses. In these cases, we have the displacement front slumped towards the bottom of the channel. The front speed is significantly faster than that of the displaced fluid directly ahead of it in the developed flow. Therefore, the streamlines are deflected upwards towards the channel center. This results in a two-dimensional flow region in which inertial effects are not negligible, although this region is being fed by upstream and downstream regions where the streamlines are pseudo-parallel. Note that the results presented in Fig. 11 were obtained for a fine mesh size of $56 \times 1000$. Qualitatively, similar results were observed in simulations run for the usual mesh size of $28 \times 400$.

4. **Overall displacing front classification**

In Sec. III D, different phenomena related to the displacement front were explained, namely, the front detachment, dispersive horns, and oscillating front speed. Now let us see where these flows are located on the dimensionless plane of $Fr$ and $Re \cos \beta/Fr$. Figure 12 shows this classification. Concentration maps added show characteristic flows in each regime. The small black circles simply indicate the values of $Fr$ and $Re \cos \beta/Fr$ for the simulations run. The first thing to note is that in a large number of simulations, front detachment has been observed. The dispersive horn-like fronts appear for $0 \leq Re \cos \beta/Fr \leq 25$, meaning near-horizontal displacements. It is now interesting to understand to what extent the oscillatory pattern in front velocity persists as the imposed
FIG. 12. Classification of our results for the full range of simulations plotted in the plane of $Fr$ and $Re \cos \beta / Fr$. The flows in which front detachment occurs are marked by $\triangle$. Data with accelerating fronts that form a dispersive horn are marked by $\triangledown$. The area $0 \leq Re \cos \beta / Fr \leq 25$, marked by a solid rectangle, roughly indicates the flows with dispersive horn at the front. The flows with oscillatory behavior in the front velocity are marked with $\triangledown$. The boundary between oscillatory and non-oscillatory flows is approximately shown by red dashed line, $Fr = 0.3$, and black solid line, $Re \cos \beta / Fr = 25$. Concentration maps added show characteristic flows with dispersive horn at front ($\beta = 90^\circ$, $Re = 50$, and $Fr = 0.19$ at $t = 5.5$), oscillatory behavior in the front velocity ($\beta = 20^\circ$, $Re = 50$, and $Fr = 0.06$ at $t = 5.5$), and front detachment ($\beta = 20^\circ$, $Re = 500$, and $Fr = 1.92$ at $t = 19.25$). The length of the domain shown is $1 \times 40$ with white solid lines indicating the position of the gate valve.

velocity, $V_0$, increases. Figure 12 shows that these flows can be observed over a large range of $Re \cos \beta / Fr$ (or equivalently inclination angles). Second, we can see that the oscillation in front velocity stops for $Fr \gtrsim 0.3$. In fact, there can be drawn an almost clear boundary between oscillatory and non-oscillatory flows in the dimensionless plane of $Fr$ and $Re \cos \beta / Fr$. It is interesting to note that all oscillatory flows also have front detachment, as shown in Fig. 12. The picture given in Fig. 12 can also be extended to the limit of exchange flows ($Fr \to 0$). In fact, the oscillating front velocity and front detachment behaviors are also found for $Re \cos \beta / Fr > 25$ in exchange flows. For $0 \leq Re \cos \beta / Fr \leq 25$, we again observe the formation of the dispersive horns at the front (see also Ref. 2 for more details on the exchange flows).

E. Macroscopic diffusion

In this section, we focus on those simulations where the degree of transverse mixing is high. In such cases, we expect that advective transport due to the mean flow will be supplemented by diffusive spreading along the pipe. Debacq et al.\textsuperscript{46,47} used a similarity scaling for exchange flows in vertical pipes to collapse profiles of the cross-sectionally averaged concentration $\bar{C}_y$, measured at each location $x$ and time $t$, onto a master curve defined with respect to $x/\sqrt{t}$. On fitting the master curve to an error function form, estimates were derived for the macroscopic diffusion of the mean concentration along the pipe. Seon et al.\textsuperscript{31} followed the same approach for inclined pipes (as have we in Ref. 32) and we adopt an analogous approach here for channel flows. Again it is worth noting that for quantitative measurement of the macroscopic diffusion coefficient, a much finer mesh is advisable. However, here we are looking only for the main qualitative trends: the large scale features and a physical interpretation of the results.

In the presence of a mean flow ($V_0$), when the flows fully mix transversely, it is logical to assume a core of the mixture traveling with the speed $V_0$ considering that the mixture diffuses axially. In this case, we might use $(x - t)/\sqrt{t}$ as a similarity scaling (see also Ref. 24). We now consider the variation in diffusive regime with different inclination angles. Figure 13 shows the collapse of depth-averaged concentration profiles $\bar{C}_y$ with $(x - t)/\sqrt{t}$.
FIG. 13. Collapse of depth-averaged concentration profiles \( \bar{C}_y \) with \((x - t)/\sqrt{t}\) for \(Re = 300\) and \(Fr = 0.61\) and (a) \(\beta = 0^\circ\), (b) \(\beta = 30^\circ\), (c) \(\beta = 60^\circ\), and (d) \(\beta = 90^\circ\). The solid line in Figs. 13(a) and 13(b) show the error-function fits with \(D_M = 1.549\) and \(D_M = 1.224\), respectively. The upper right insets show the qualitative flow pattern in each case. In part (d), the lower left inset shows the collapse of the concentration profiles when \(x/t\) is used instead of \((x - t)/\sqrt{t}\).

It can be seen in Fig. 13 that when the mixing is very strong and efficient (\(\beta = 0^\circ\)), the collapsed profiles fall onto an almost exact error-function shape. Waves and slight deviations from the fitted error function curve are due to short time behavior and the initial stages of mixing. At substantially longer times, we perhaps will observe a smoother fit for transversely mixed flows. For each case (shown in Fig. 13), at least 30 profiles are used with time steps of \(dt = 0.95\) when the flow is most developed \((t \geq 12)\). As we move away from the strictly vertical channel (weaker mixing, more slumping), the collapsed profiles start to deviate from the diffusive flow fit function. This deviation appears mostly in the form of a drop in concentration values \(\bar{C}_y\) for \((x - t)/\sqrt{t} < 0\), meaning the region upstream the mixing core (see Figs. 13(b) and 13(c)). The solid lines in Figs. 13(a) and 13(b) show a curve fit of the form \(\bar{C}_y = 0.5erfc\left(\frac{x - t}{2\sqrt{D_M}t}\right)\), motivated by the solution to the linear diffusion equation. Here, the macroscopic diffusion coefficient has been made dimensionless as \(D_M = \hat{D}_M/(\hat{V}_0\hat{D})\). The deviation from the symmetric error function fit means that the mixing downstream and upstream of the mixing core has different qualities. In fact, in most of the simulations where strong transverse mixing is observed, the downstream zone of the mixed core was found to be more diffusive than the upstream region. In other words, mixing close to the leading front is stronger than that close to the trailing front (see Fig. 13 for \(\beta = 30^\circ\) or \(\beta = 60^\circ\), for instance). For the (nearly) horizontal cases where the flow is more segregated at the interface, the concentration profiles do not collapse well onto an error function fit at all, but are more advective; see Fig. 13(d). In this case, a more sensible similarity scaling to use is \(x/t\) rather than \((x - t)/\sqrt{t}\), as shown in the inset to Fig. 13(d).

We now focus only on those simulations for which we have been able to reliably collapse the data at long times, using the similarity scaling \((x - t)/\sqrt{t}\), and have been able to estimate a macroscopic diffusion coefficient from the master curve. We call these flows truly diffusive.
dimensional macroscopic diffusion coefficient, $D_M$, measured from the fully diffusive simulations varies in the range of $5 \times 10^{-5} - 1 \times 10^{-3}$ (m$^2$/s) which is much higher than the fully turbulent diffusivity in the same range of $Re$ (see also Ref. 32 for similar effect in pipe displacement flow).

The relatively large value of macroscopic diffusion coefficient confirms that the axial diffusion that we observe is due to advection of the concentration, i.e., dispersion, via a process analogous to Taylor dispersion but with the local mixing driven by buoyancy rather than shear-driven turbulence. The flows are disordered and relatively well-mixed transversely, to all appearances locally turbulent. In such cases, a scaling of the dispersivity with length and velocity scales is expected that are relevant to the eddy structure. This is evidently complex in general, but we might guess that the relevant velocity will scale approximately with either the inertial velocity or the mean flow, depending on the relative strengths of buoyancy or the mean flow in driving the mixing.

To explore this, we plot $\hat{D}_M/(\hat{V}_l\hat{D})$ and $\hat{D}_M/(\hat{V}_b\hat{D})$ for all fully diffusive experiments in the $(Fr, Re \cos \beta/\hat{F}r)$-plane (see Fig. 14), the two axes reflecting the competition between inertia from the mean flow and buoyancy ($Fr$) and between buoyancy driven motion and viscous dissipation of that motion ($Re \cos \beta/\hat{F}r$), respectively. Concentration maps added show characteristic diffusive and non-diffusive flows. Note that $\hat{V}_l$ is a velocity scale obtained by the balance between inertia and buoyancy ($\hat{V}_l = \sqrt{At \hat{g} \hat{D}}$). We can see that scaling of $\hat{D}_M$ suggests that both mechanisms are responsible for the mixing in different limits. For small $Fr$, at large ($Re \cos \beta/\hat{F}r$) as we transition out of the exchange dominated regime see that $\hat{D}_M/(\hat{V}_l\hat{D})$ appears to become independent of $Fr$, suggesting buoyancy driven mixing. At the other extreme, for modest ($Re \cos \beta/\hat{F}r$) and as we increase $Fr$, we find $\hat{D}_M/(\hat{V}_b\hat{D}) \approx$ constant, suggesting mixing driven by the mean flow. Upon comparison with the experiments in pipe, it can be noted that due to the more unstable nature of the flow in channel displacement over the pipe, fully diffusive flows can also be found for smaller $Re \cos \beta/\hat{F}r$ (or equivalently $At$). Also it is checked that the dimensional macroscopic diffusion coefficient, $D_M$, increases mean flow speed, $\hat{V}_b$, similar to the experiments in pipe. The results however are not shown here in order to keep the consistency of presenting everything in dimensionless format. In the case of pipe displacement, it was also observed that $\hat{D}_M$ increases with $\beta$, an effect which is only seen at the highest density difference ($At = 0.01$) in the channel simulations. For moderate and low density differences, a general trend of increase or decrease in $\hat{D}_M$ with $\beta$ cannot be drawn.

IV. DISCUSSION AND CONCLUSIONS

In Sec. III, we presented our results in terms of the velocity of the leading displacement front and in terms of bulk axial diffusivity in the case of fully diffusive flows. We now try to give an overall qualitative description of the different flow regimes observed at long times in our simulations, in terms of the 3 dimensionless parameters: $(Fr, Re, \beta)$. Much of the parametric variation appears
FIG. 15. Classification of our results for the full range of simulations plotted in the plane of $Fr$ and $Re \cos \beta / Fr$. The instantaneous displacement flows are colored in blue (darker gray) and non-instantaneous ones in red (lighter gray). The thick dashed lines show the boundary between diffusive and non-diffusive flows. The horizontal dashed lines are drawn at $Re \cos \beta / Fr = 350$ and $Re \cos \beta / Fr = 180$ and the vertical dashed line is at $Fr = 1.1$. Non-diffusive flows are marked by either the superposed circles (viscous flows) or squares (intermittent flows). The transition between viscous and inertial regimes is re-confirmed to occur at $Re \cos \beta / Fr = 25$ marked by a horizontal solid line. Concentration maps added show characteristic diffusive ($\beta = 0^\circ$), intermittent ($\beta = 80^\circ$), and viscous ($\beta = 90^\circ$) flows. The rest of the parameters used in the concentration maps are $Re = 300$, $Fr = 0.61$, and $t = 16.5$. The length of the domain shown is $1 \times 40$. White solid lines indicate the position of the gate valve.

Figure 15 shows the numerical points from our study, characterized in terms of flow type. We distinguish flows according to the following criteria: (i) instantaneous displacement (if there is no displaced fluid observed above the gate valve position); (ii) fully diffusive (if we are able to collapse the data at long times via a similarity solution of the form, $\bar{C}_y = 0.5erfc\left(\frac{x}{2\sqrt{D_Mt}}\right)$); (iii) non-diffusive flows, we classify as either viscous flows if there is no instability evident in the spatiotemporal image behind the leading displacement front and inertial otherwise. It can be seen that this classification, although not perfect, does appear to separate the data within the $(Fr, Re \cos \beta / Fr)$-plane. Examples of characteristic flows in each regime are added to Fig. 15 for better understanding.

The first thing to note is that compared to the pipe displacement flow results in Ref. 32, viscous flows in the 2D channel are found over a rather narrow range $Re \cos \beta / Fr < 25$. The boundary between the diffusive and non-diffusive flows can be drawn using $Re \cos \beta / Fr = 350$ for $Fr < 1.1$ and $Re \cos \beta / Fr = 180$ for $Fr > 1.1$ (Fig. 15). For instance, the viscous flows found seem to remain viscous on increasing $Fr$ number (or increasing $V_0$ equivalently). The inertial non-diffusive flows (intermittent flows) within the range $25 < Re \cos \beta / Fr < 180$ also stay unaffected by the imposed flow. For data with $Re \cos \beta / Fr \geq 350$, the flows are already diffusive for even small $Fr$ numbers (or small $V_0$ values). Increasing $Fr$ in this range does not change the qualitative characteristics of the flow. Note however that the range of Reynolds number chosen in this study is limited to $0 < Re < 800$, so that the usual effects of inertially driven shear instability at higher Reynolds numbers (equivalently higher Froude numbers) might not manifest in the range chosen.

The only range of $Re \cos \beta / Fr$ in which the imposed flow seem to have an effect on the flow stability is $180 < Re \cos \beta / Fr < 350$. In this range, the inertial non-diffusive flows transform into fully diffusive flows at $Fr = 1.1$. Note that although the influence of the imposed flow on flow stability is not very pronounced in the case of the 2D channel flows studied, it still has a large impact on making displacement flows transition to instantaneous displacements from non-instantaneous displacements. Note the non-instantaneous (red) data points that transform to instantaneous flows (blue) under the influence of increasing $Fr$. The flow regimes defined in Fig. 15 can also be extended to the limit of exchange flows, $Fr \to 0$.

To summarize, displacement flow of two miscible iso-viscous Newtonian fluids in an inclined 2D channel has been investigated numerically in the case where the displacing fluid is denser than...
the displaced fluid (i.e., density unstable). Our simulations have covered a broad range of the governing dimensionless parameter space ($\beta, Re, Fr$), not covered before in any numerical study. The qualitative features of the flow at different inclination angles were investigated through concentration field maps, spatiotemporal diagrams of the averaged concentration, and also the contours of the velocity field. As we move on closer to vertical and higher inclination angles, mixing and instability increase within the flow.

Rayleigh-Taylor mushroom-like structures were found for strictly vertical 2D channels, only for relatively low imposed flow rate and density difference. Due to both mixing and the imposed flow, the mushroom-like structure is not sustained over longer times. For inclinations slightly away from the vertical, the mushroom pattern was not observed due to the effects of slumping, i.e., symmetry is broken in the initial onset of flow, even though $\beta$ is very small.

Similar to the pipe flow experiments for flow design, the two most important quantities to measure in 2D channel displacement flows are front velocity and macroscopic diffusion coefficient. The latter was measured only for fully diffusive flows. In the case of the pipe displacement, when the mixing is efficient, the front velocity value is found to be very close to that of the mean flow, i.e., $V_f \approx V_0$. However, for 2D channel flow even when the fluids mix significantly, the tip of the displacing front still advances close to the channel center, resulting in $V_f \approx 1.5V_0$ (by analogy with the Poiseuille flow). Potentially, if the simulations were run over a longer channel length and for longer times, the mixing could have been more complete transversely resulting in $V_f \approx V_0$.

Different interesting patterns were observed at the tip of the displacing front, namely, front detachment and dispersive horn-like fronts. These phenomena have been explained physically and have been located on dimensionless flow regime maps. Similarly, we took a similar approach with exchange dominated flows that exhibited some oscillatory-type behavior in the front velocity. These flows were found over the range of $Re \cos(\beta)/Fr > 25$ and for $Fr < 0.3$. Finally, we have classified flows as fully diffusive, instantaneous, inertial, and viscous, providing a qualitative description of each and delineating where each flow can be found in the dimensionless planes of $Fr$ and $Re \cos(\beta)/Fr$.

ACKNOWLEDGMENTS

This research has been carried out at the University of British Columbia, supported financially by NSERC and Schlumberger through CRD Project No. 444985-12. The research was also enabled in part by support provided by WestGrid (www.westgrid.ca) and Compute Canada (www.computecanada.ca). The help of Dr. K. Wielage-Burchard in validating the numerical code used is highly appreciated. The authors thank the reviewers for their helpful and constructive comments in improving the quality of the current study.
